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DUFFING–HARMONIC OSCILLATOR**

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**Abstract:** *This paper presented a more efficient differential transform method (DTM) for solving the Duffing–harmonic oscillator. The proposed method significantly minimizes computational effort while maintaining a high level of accuracy in capturing the nonlinear dynamic behavior of the system. The present scheme employs an algebraic series solution that avoids trigonometric functions, thereby reducing computational complexity. Through a detailed comparison study, the performance of the improved DTM was evaluated against a very powerful global residue harmonic balance method (GRHBM) and numerical techniques, demonstrating its effectiveness in generating reliable approximate solutions for Duffing–harmonic oscillators. Thus, the results highlight the method’s capability to handle strong nonlinearities without requiring complex iterative schemes or heavy computational resources. Overall, the findings indicate that the developed DTM serves as a valuable and powerful tool for advancing the analysis of nonlinear dynamic systems and can be readily extended to a wide range of related nonlinear oscillatory models.*

**Subject Classification:** (2010) 34A34, 34B15, 34C15, 34G20

**Keywords:** *Duffing harmonic oscillator; Approximate analytic technique; Differential transform method (DTM); Global residue harmonic balance method (GRHBM).*

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## 1. Introduction

The Duffing harmonic oscillator is a nonlinear second-order differential equation that several scientists [1-5] have theoretically worked on because of its importance and application in the fields of physics, engineering, and applied mathematics. The equation describes many physical systems, such as those that create mechanical vibration, heat and cool and electrical circuits. Due to its strong nonlinear behavior, it is often difficult to achieve at exact analytical solutions to the Duffing oscillator, especially to determine answers to high amplitude oscillations or complicated boundary

conditions. Therefore, significant academic research has been channeled into the development of approximate and semi-analytical methods that can provide accurate, effective, and practicable solutions.

Many analytical methods have been introduced for solving the Duffing oscillator, which include perturbation methods [6-8], the harmonic balance method (HBM) [9-14], the homotopy perturbation method (HPM) [15-16] and the energy balance method (EBM) [17-18]. As much as these methods can be used to determine rough periodic solutions, their accuracy and convergence rate often become worse when used on highly nonlinear systems. Moreover, higher-order approximations are often subject to numerous and complicated manipulations of algebraic terms. To overcome these limitations, advanced models, such as the global residue harmonic balance method (GRHBM) [re14], were developed, thus leading to better accuracy in solutions by summing up the cumulative error. However, the methods are still computationally intensive and there are problems generalizing them to more general nonlinear systems.

Over the last few years, the Difference Transform Method (DTM), first proposed by Zhou [19], has received attention as a simple but effective semi-analytical method for solving nonlinear differential equations. The differential transform method (DTM) which transforms differential equations into a system of algebraic recurrence relations, reduces computational efforts without decreasing the accuracy of the solutions. Many researchers have used DTM to obtain approximate solutions for various scientific problems. This method has been explored for solving systems of differential equations [20], duffing harmonic oscillators [6], duffing oscillator equations with damping effects [21], relativistic oscillator [19] and several nonlinear oscillators [22-24]. Generally, accurate and acceptable solutions are obtained for the initial position through the application of this semi-analytic approach. An approximate solution in a finite series form about the initial time has been found to be of great help by the DTM. The development of MDTM [25] has overcome the limitations of DTM.

A more effective formulation of a Differential Transform Method (DTM) is introduced in this study to solve the Duffing harmonic oscillator. The presented method provides better accuracy and at the same time decreases the complexity of the computation and therefore converges faster compared to the other existing methods, such as GRHBM [14]. The results prove that the presented

DTM is an effective and reliable tool in the study of nonlinear oscillatory systems because it offers analytical simplicity to the calculation as well as computational efficiency throughout a large range of amplitudes.

## 2. Present Method

Consider the nonlinear differential oscillator's equation

$$\ddot{x} + \omega_0^2 x + \varepsilon f(x, \dot{x}) = 0 \quad \text{with } x(0) = a \text{ and } \dot{x}(0) = 0 \quad (1)$$

where the over dot indicates difference with regard to  $t$ ,  $\omega_0$  has been used to represent the natural frequency of the system,  $f(x, \dot{x})$  is a nonlinear function and  $\varepsilon$  is a small perturbation parameter.

Let us consider a polynomial-type series expansion for  $x(t)$  is presented as follows

$$x(t) = a + B_2 t^2 + B_4 t^4 + B_6 t^6 + B_8 t^8 + B_{10} t^{10} + B_{12} t^{12} + \dots \quad (2)$$

Where  $B_2, B_4, B_6, B_8, B_{10}, B_{12}$  etc. represent unknown coefficients that need to be determined.

It is clear that Eq. (2) satisfies the first initial condition as it is expressed in Eq. (1).

The series considered as Eq. (2) is substituted in the governing Eq. (1) and then equating the coefficients of similar powers of  $t$  on both sides there generates a system of linear algebraic equations. Now finding the solution of this system of algebraic equations, one can determine the values of  $B_2, B_4, B_6, B_8, B_{10}, B_{12}$  and so on.

Now, putting the values of  $B_2, B_4, B_6, B_8, B_{10}, B_{12}$  etc. into Eq. (2), then the solution of the oscillator is obtained.

## 3. Example

Consider a Duffing harmonic oscillator whose equation takes the form

$$\ddot{x} + \frac{x^3}{1+x^2} = 0, \quad \text{with } x(0) = a \text{ and } \dot{x}(0) = 0 \quad (3)$$

where the over dot indicates difference with regard to  $t$ .

Eq. (3) can be written as

$$\ddot{x}(1 + x^2) + x^3 = 0, \quad \text{with } x(0) = a \text{ and } \dot{x}(0) = 0 \quad (4)$$

Let us consider a polynomial-type series expansion for  $x(t)$  is presented as follows

$$x(t) = a + B_2t^2 + B_4t^4 + B_6t^6 + B_8t^8 + B_{10}t^{10} + B_{12}t^{12} + \dots \dots \dots \quad (5)$$

Where  $B_2, B_4, B_6, B_8, B_{10}, B_{12}$  etc. represent unknowns to be determined.

By differentiating Eq. (5) with respect to  $t$  and then replacing the terms  $\ddot{x}(t)$  and  $x(t)$  into Eq. (4), we obtain,

$$(2B_2 + 12B_4t^2 + 30B_6t^4 + 56B_8t^6 + 90B_{10}t^8 + 132B_{12}t^{10} + \dots \dots \dots) + (1 + (a + B_2t^2 + B_4t^4 + B_6t^6 + B_8t^8 + B_{10}t^{10} + B_{12}t^{12} + \dots \dots \dots)^2) + (a + B_2t^2 + B_4t^4 + B_6t^6 + B_8t^8 + B_{10}t^{10} + B_{12}t^{12} + \dots \dots \dots)^3 = 0 \quad (6)$$

Now, equating the like power of  $t$ , we get systems of linear equations

$$\left(\frac{a^3}{1+a^2} + 2B_2\right) = 0 \quad (7)$$

$$\left(\frac{a^2(3+a^2)B_2}{(1+a^2)^2} + 12B_4\right) = 0 \quad (8)$$

$$\left(\frac{(3aB_2^2 - a^3B_2^2 + 30B_6 + 3a^2(B_4 + 30B_6) + a^6(B_4 + 30B_6) + a^4(4B_4 + 90B_6))}{(1+a^2)^3}\right) = 0 \quad (9)$$

By solving Eqs. (7), (8) and (9) then we get unknown coefficients

$$B_2 = -\frac{a^3}{2(1+a^2)}, \quad B_4 = \frac{a^5(3+a^2)}{24(1+a^2)^3}, \quad B_6 = -\frac{a^7(27+a^4)}{720(1+a^2)^5}.$$

Similarly, we can get the values of  $B_8, B_{10}, B_{12}$  and so on.

Now, putting the values of  $B_2, B_4, B_6$ , etc. into Eq. (5) then we obtain

$$x(t) = a - \frac{a^3t^2}{2(1+a^2)} + \frac{a^5(3+a^2)t^4}{24(1+a^2)^3} - \frac{a^7(27+a^4)t^6}{720(1+a^2)^5} + \frac{a^9(441-513a^2+63a^4+a^6)t^8}{40320(1+a^2)^7} - \frac{a^{11}(11529-36000a^2+21054a^4-1320a^6+a^8)t^{10}}{3628800(1+a^2)^9} + \dots \dots \dots \quad (10)$$

Which is the required solution.

## 4. Results and Discussions

The present differential transform method (DTM) has accurately reproduced the nonlinear oscillator periodic response as it appears in all the amplitudes investigated. The comparison with a fourth-order Runge-Kutta (RK-4) and the global residual harmonic balance method (GRHBM) demonstrates that even for small oscillations, DTM has already been quite competitive and tends to become even more accurate with the increase in amplitude. The frequency errors of

DTM decrease with amplitude at the same rate as those of GRHBM, while the decay rates of DTM's errors match those of GRHBM; thus, the results show a slight benefit for GRHBM at small amplitudes and a significant benefit for DTM at moderate to large amplitudes.

In Table 1, a summary of the frequency comparison is given. GRHBM at the lowest amplitude,  $a = 1$ , yields a slightly lower error than DTM; however, when  $a > 1$ , the frequency error of DTM is always smaller, and at  $a = 10$  and  $50$ , it is almost negligible. This observation indicates that the DTM reoccurrence is effective to extract the leading nonlinear stiffness terms and eliminates the need for bookkeeping residual terms and additional harmonic balancing terms, which increases the algebraic cost of the GRHBM in large amplitude regimes. Practically, DTM is able to obtain almost perfect fundamental frequencies using a small series order, and it is therefore beneficial in parametric studies that require quick results.

In Figs. 1, 3, and 5 show solution responses with the help of the time domain. The solutions are obtained using the present method, as well as RK-4 and GRHBM. In the case of  $a = 1$  (Fig. 1), the DTM solution fits the RK-4 solution nearly at a number of periods and there is a small phase difference between the two at the turning points; the GRHBM gives an equally good fit. At the intermediate value of  $a = 5$  (Fig. 3), the DTM curve and RK-4 nearly coincide throughout the entire cycle and GRHBM shows a slight but noticeable phase drift. Even when the amplitude is huge (Fig. 5), the DTM is still able to follow the RK-4 signal with high precision, thus indicating that it is quite robust even in conditions where the nonlinearity is severe.

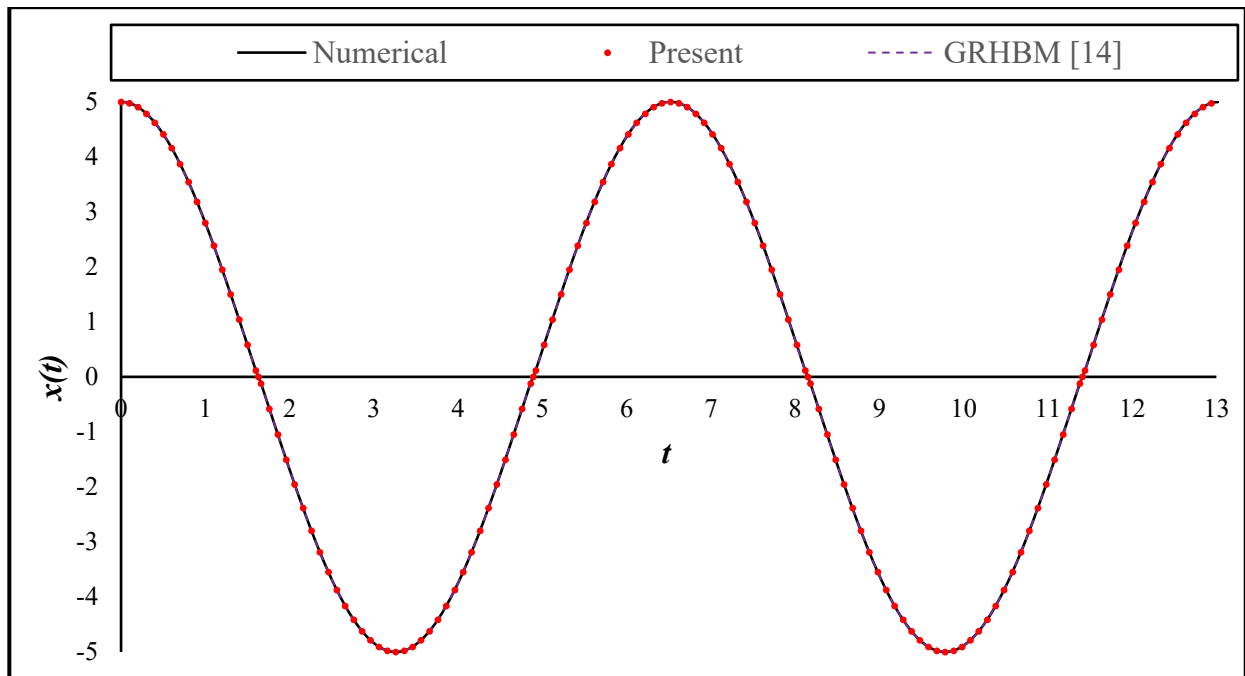
The error analyses are as shown in Figs. 2, 4, and 6. The errors are calculated by the equation  $error = |Numerical\ value - approximate\ values|$ .

In Fig. 2, for amplitude  $a = 5$ , it is shown that DTM has slightly smaller error values than GRHBM in the oscillation cycle, thus confirming its higher accuracy and convergence in amplitude levels. In Fig. 4, for a higher amplitude  $a = 10$ , the DTM has very high accuracy and the error values are almost negligible compared with GRHBM. This contract proves the strength of the DTM when the nonlinearities are strong. In Fig. 6, in the case of very large amplitude  $a = 100$ , the DTM shows almost zero relative error performance, which is much better in comparison with the GRHBM. This illustrates that the presented DTM is able to handle a great amount of nonlinearity without losing computational simplicity and stability.

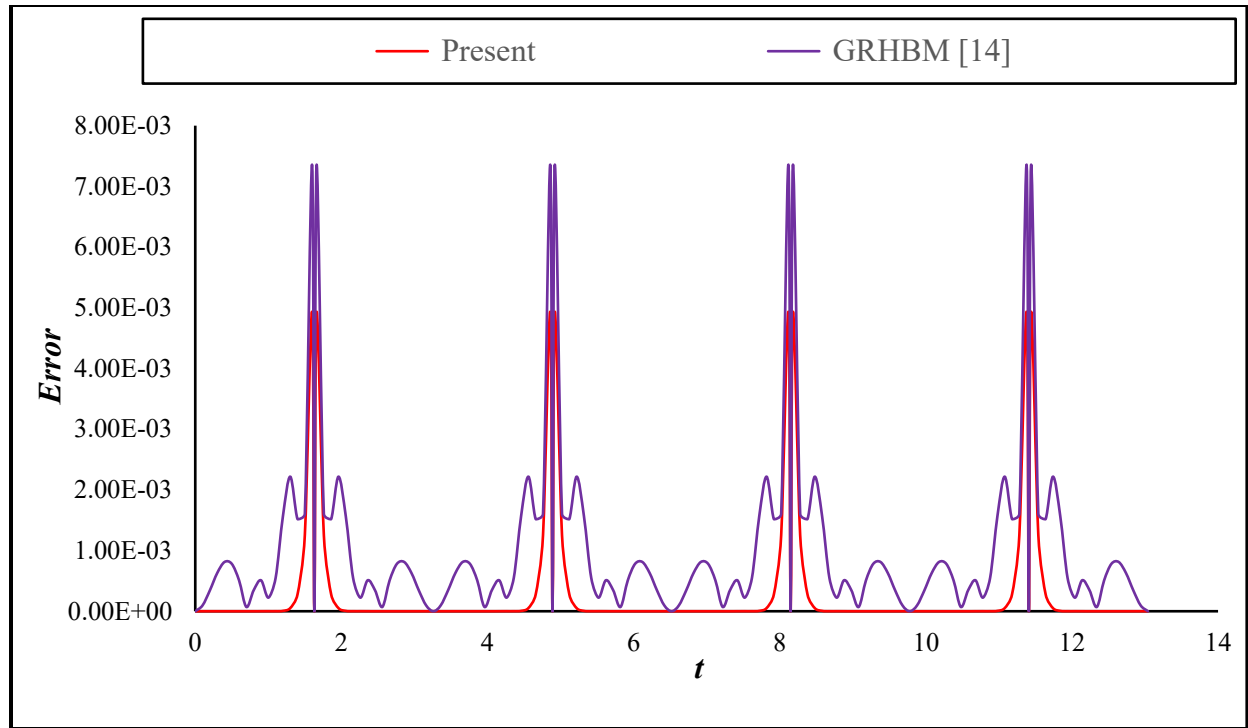
**Table 1:** Comparison of the present frequencies with exact values and other existing frequencies for duffing harmonic oscillator.

Amplitude <b>a</b>	Frequencies $\omega$			Error (%)	
	Numerical	GRHBM [14]	Present Method	GRHBM [14]	Present Method
1	0.636780	0.636795	0.638838	0.002355	0.323188
5	0.966975	0.968107	0.966105	0.117066	0.089971
10	0.990916	0.991591	0.991090	0.068118	0.017559
50	0.999607	0.999657	0.999638	0.005001	0.003101
100	0.999900	0.999914	0.999909	0.001400	0.000900

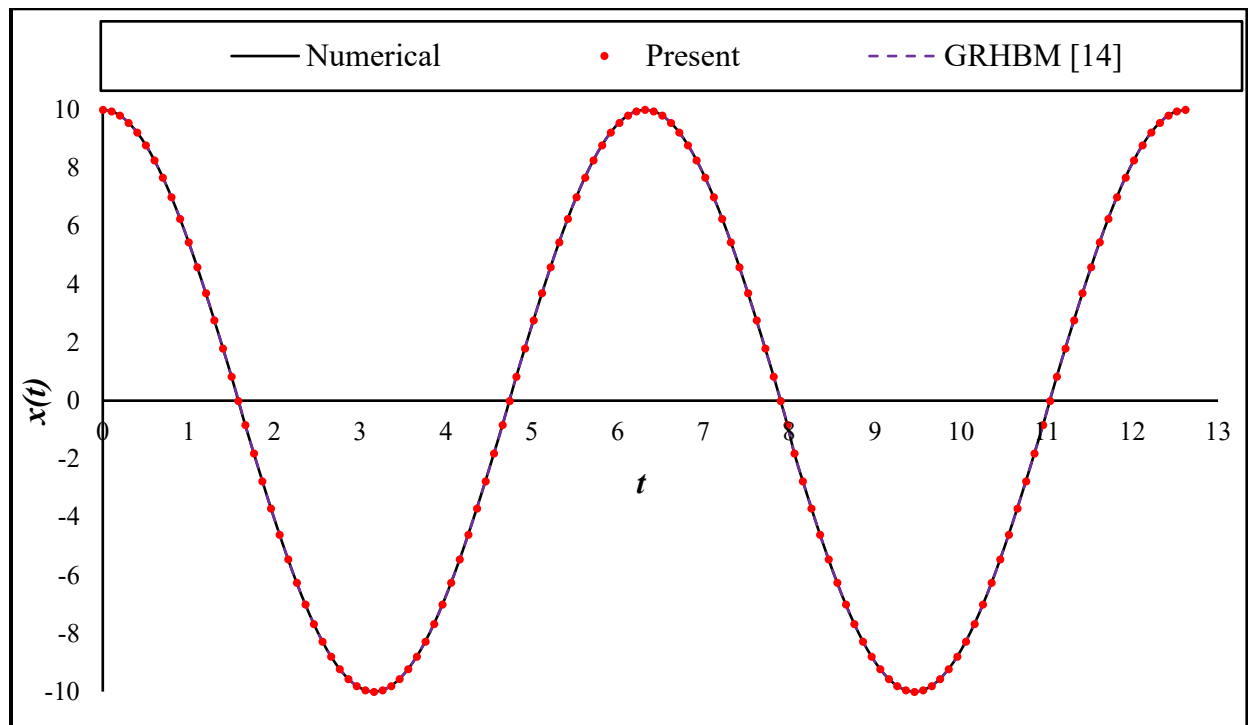
Where, Error (%) denotes the absolute percentage error.



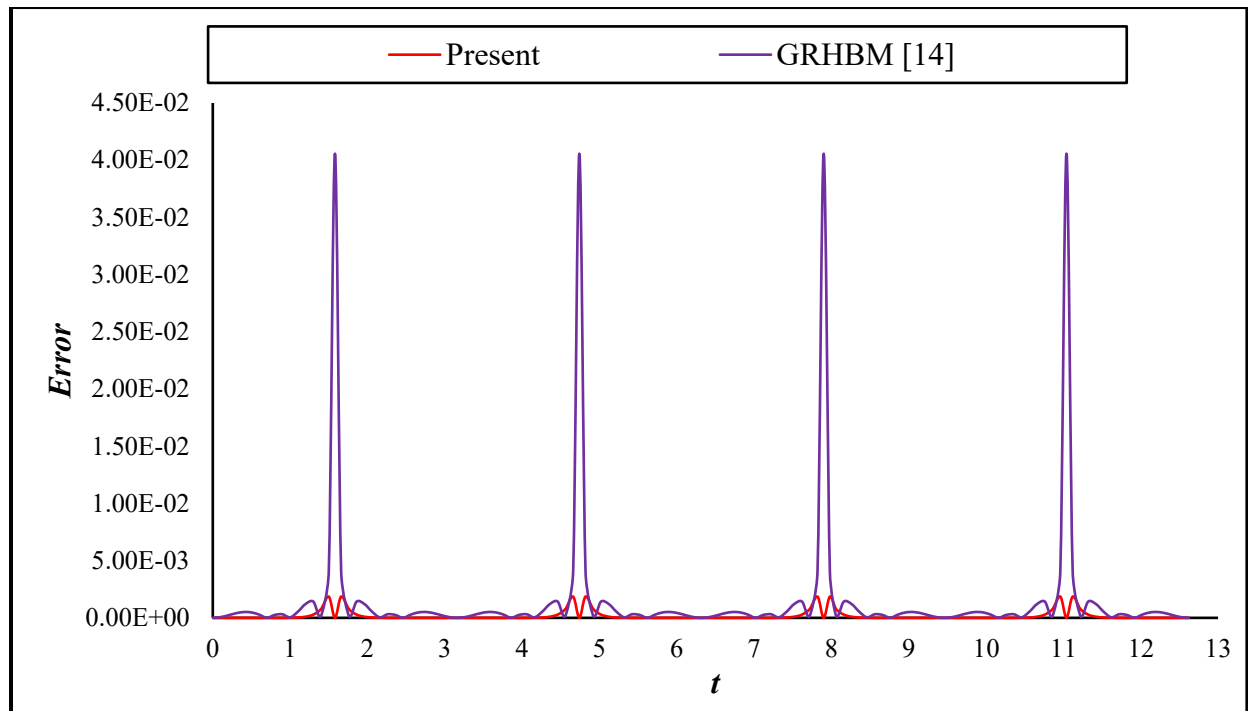
**Figure 1:** Comparison among the present solution with numerical solution and GRHBM [14] solution for  $a = 5$



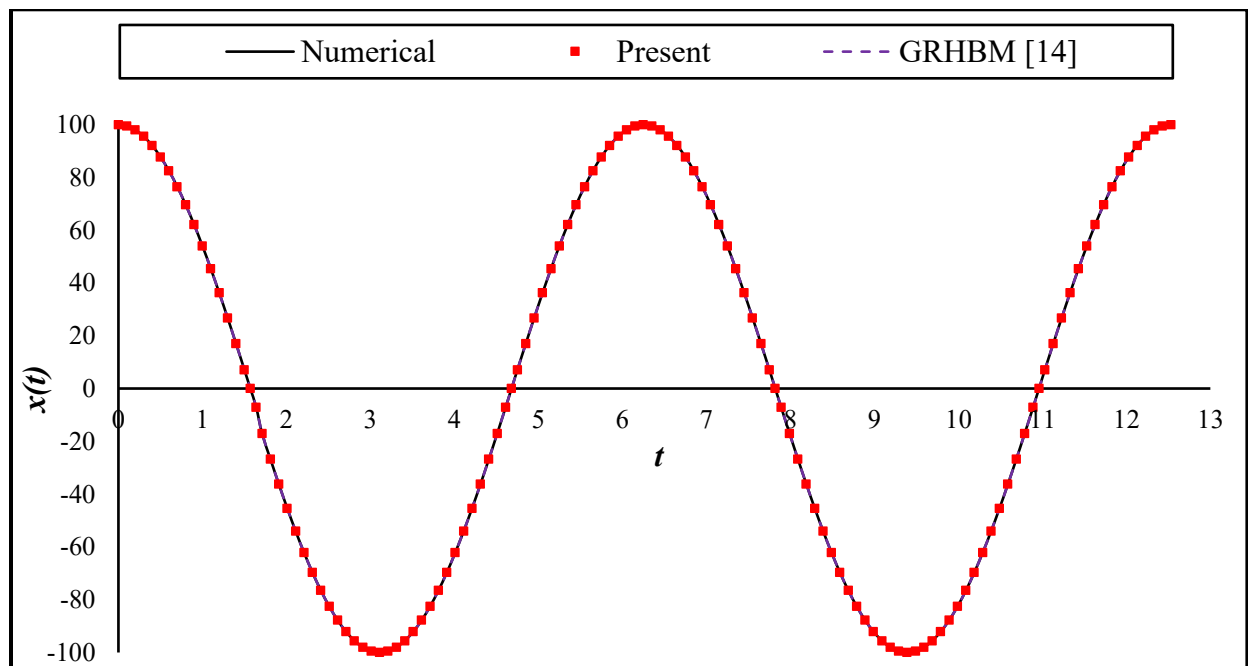
**Figure 2:** Comparison of the present error with GRHBM [14] error for  $a = 5$



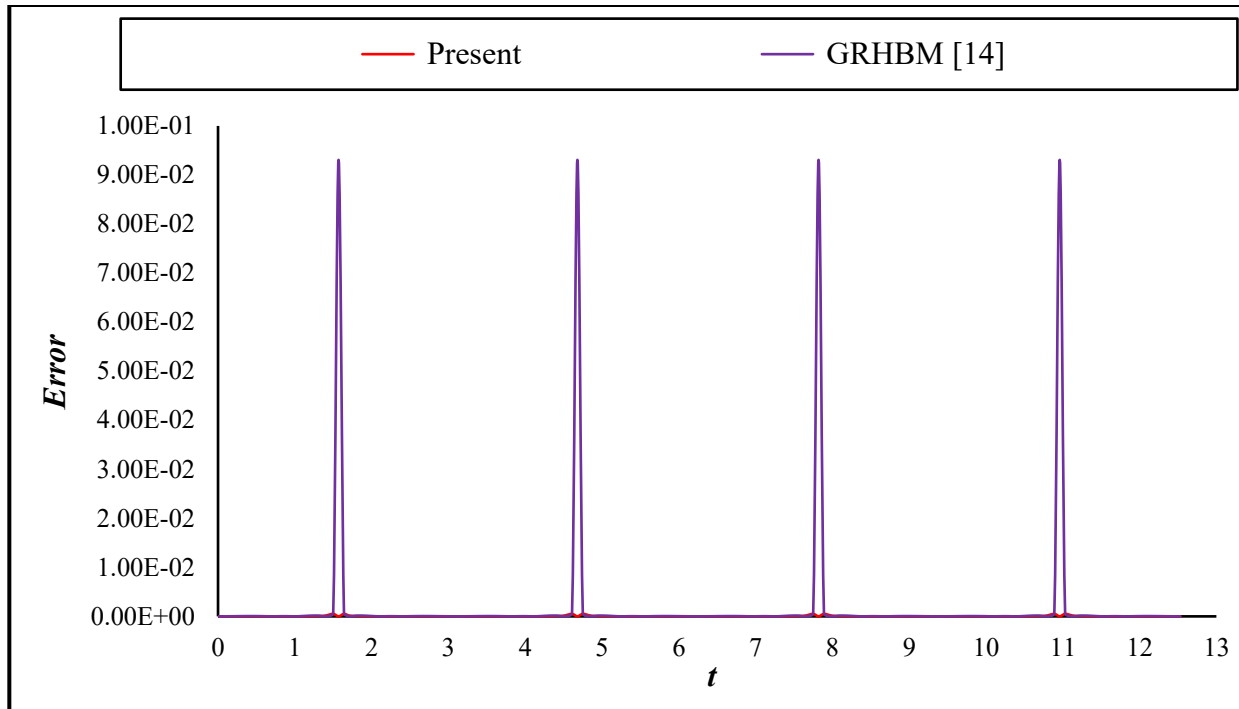
**Figure 3:** Comparison among the present solution with numerical solution and GRHBM [14] solution for  $a = 10$



**Figure 4:** Comparison of the present error with GRHBM [14] error for  $\alpha = 10$



**Figure 5:** Comparison among the present solution with numerical solution and GRHBM [14] solution for  $\alpha = 100$



**Figure 6:** Comparison of the present error with GRHBM [14] error for  $a = 100$

In general, the findings demonstrate that the differential transformation method is an easy-to-use and high-quality tool for strongly nonlinear periodic oscillations. It achieves higher accuracy when operating on large amplitudes and generates less algebraic overhead compared to the GRHBM, and it maintains the performance of small amplitudes that are, however, competitive with existing benchmarks.

## 5. Conclusion

The Duffing-harmonic oscillator, which can be characterized by its high level of nonlinearity and broad practical applicability in both physics and engineering, is not an easy object to analyze. In the current paper, a more effective Differential Transform Method (DTM) is introduced to solve the Duffing harmonic oscillator equation. The aim of the presented method is to obtain a higher level of accuracy and lower the number of required computations. Through the presented method, nonlinear dynamic responses were effectively analyzed, which showed an outstanding consistency over the Global Residue Harmonic Balance Method (GRHBM) [14] as well as with the fourth-order Runge-Kutta (RK-4) numeric technique.

The findings prove that the present DTM ensures high-ranked convergence and also provides a more accurate assessment of oscillation frequencies in a wide scale of amplitude levels. With small amplitude oscillations, the current DTM can perform as well as GRHBM [14]; however, at large amplitude, the technique does significantly better than GRHBM [14] in terms of accuracy and efficiency. The present Differential Transformation Method (DTM) reduces algebraic manipulations and separates important nonlinear stiffness terms without the need to correct the residual error, making it a more convenient and economical method of computation. Moreover, the frequency errors were negligible, and the amplitude-frequency relationship obtained using DTM was in close comparison with the numerical standard, thus confirming the robustness of the method. Overall, the presented DTM formulation provides a powerful, accurate and economical analytical format and thus further develops the research of nonlinear dynamic systems and opens a bright future for the further progress of applied mechanics and vibration analysis.

### **Author Contribution**

Mostafa Amir Faishal: investigation; validation; writing—original draft; writing—review & editing.

Nazmul Sharif: formal analysis; investigation; methodology; resources; validation; writing—original draft; writing—review & editing.

M. Kamrul Hasan: Conceptualization; formal analysis; investigation; methodology; resources; validation; writing—original draft; writing—review & editing.

### **Conflict of Interest**

No conflict-of-interest present among the authors.

### **Data Availability**

No datasets were generated or analyzed during the current study.

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## References

- [1] M. Aktasa, B. Nayakb, and B. Rathc, “Exact analytical investigation of Duffing oscillator vibration under time-periodic oscillatory external force,” *J. Energy*, vol. 4, no. 3, 2024.
- [2] G. M. Moatimid, “Stability analysis of a parametric Duffing oscillator,” *J. Eng. Mech.*, vol. 146, no. 5, Art. no. 05020001, 2020.
- [3] A. H. Salas Salas, J. E. Castillo Hernández, and L. J. Martínez Hernández, “The Duffing oscillator equation and its applications in physics,” *Math. Probl. Eng.*, vol. 2021, no. 1, Art. no. 9994967, 2021.
- [4] N. A. Khan, T. Hameed, O. A. Razzaq, and M. Ayaz, “Intelligent computing for Duffing-harmonic oscillator equation via the bio-evolutionary optimization algorithm,” *J. Low Freq. Noise Vib. Act. Control*, vol. 38, nos. 3–4, pp. 1327–1337, 2019.
- [5] S. S. Ganji, D. D. Ganji, A. G. Davodi, and S. Karimpour, “Analytical solution to nonlinear oscillation system of the motion of a rigid rod rocking back using max–min approach,” *Appl. Math. Model.*, vol. 34, no. 9, pp. 2676–2684, 2010.
- [6] M. K. Abohamer, T. S. Amer, A. Arab, and A. A. Galal, “Analyzing the chaotic and stability behavior of a Duffing oscillator excited by a sinusoidal external force,” *J. Low Freq. Noise Vib. Act. Control*, vol. 44, no. 2, pp. 969–986, 2025.
- [7] G. M. Ismail, M. Abul-Ez, M. Zayed, H. Ahmad, and M. El-Moshneb, “Highly accurate analytical solution for free vibrations of strongly nonlinear Duffing oscillator,” *J. Low Freq. Noise Vib. Act. Control*, vol. 41, no. 1, pp. 223–229, 2022.
- [8] M. S. Alam, N. Sharif, and M. H. U. Molla, “Combination of modified Lindstedt–Poincaré and homotopy perturbation methods,” *J. Low Freq. Noise Vib. Act. Control*, vol. 42, no. 2, pp. 642–653, 2023.
- [9] U. Von Wagner, L. Lentz, H. Dänschel, and N. Gräbner, “On large amplitude vibrations of the softening Duffing oscillator at low excitation frequencies—Some fundamental considerations,” *Appl. Sci.*, vol. 14, no. 23, Art. no. 11411, 2024.

- [10] M. I. Qaisi, “Solution of Duffing-harmonic oscillator by the power series method,” *J. Mech. Eng. Autom.*, vol. 11, no. 6, pp. 165–170, 2021.
- [11] R. E. Mickens, “Mathematical and numerical study of the Duffing-harmonic oscillator,” *J. Sound Vib.*, vol. 244, no. 3, pp. 563–567, 2001.
- [12] H. Hu and J. H. Tang, “Solution of a Duffing-harmonic oscillator by the method of harmonic balance,” *J. Sound Vib.*, vol. 294, no. 3, pp. 637–639, 2006.
- [13] A. Beléndez, D. I. Méndez, E. Fernández, S. Marini, and I. Pascual, “An explicit approximate solution to the Duffing-harmonic oscillator by a cubication method,” *Phys. Lett. A*, vol. 373, no. 32, pp. 2805–2809, 2009.
- [14] P. Ju, “Global residue harmonic balance method for Helmholtz–Duffing oscillator,” *Appl. Math. Model.*, vol. 39, no. 8, pp. 2172–2179, 2015.
- [15] N. Anjum and J. H. He, “Two modifications of the homotopy perturbation method for nonlinear oscillators,” *J. Appl. Comput. Mech.*, vol. 6, Special Issue, pp. 1420–1425, 2020.
- [16] C. H. He and Y. O. El-Dib, “A heuristic review on the homotopy perturbation method for non-conservative oscillators,” *J. Low Freq. Noise Vib. Act. Control*, vol. 41, no. 2, pp. 572–603, 2022.
- [17] M. A. Hosen, G. Ismail, A. Yildirim, and M. A. S. Kamal, “A modified energy balance method to obtain higher-order approximations to the oscillators with cubic and harmonic restoring force,” *J. Appl. Comput. Mech.*, vol. 6, no. 2, pp. 320–331, 2020.
- [18] S. V. Kontomaris, G. Chliveros, and A. Malamou, “Approximate solutions for undamped nonlinear oscillations using He’s formulation,” *J*, vol. 6, no. 1, pp. 140–151, 2023.
- [19] M. A. Huq, M. K. Hasan, and M. S. Alam, “An approximation technique for solving nonlinear oscillators,” *J. Low Freq. Noise Vib. Act. Control*, vol. 43, no. 1, pp. 239–249, 2024.
- [20] R. Saadeh, A. A. Alshawabkeh, R. Khalil, M. A. Abdoon, N. Taha, and D. K. Almutairi, “The Mohanad transforms and their applications for solving systems of differential equations,” *Eur. J. Pure Appl. Math.*, vol. 17, no. 1, pp. 385–409, 2024.
- [21] N. H. Aljahdaly and S. A. El-Tantawy, “On the multistage differential transformation method for analyzing damping Duffing oscillator and its applications to plasma physics,” *Mathematics*, vol. 9, no. 4, Art. no. 432, 2021.

- [22] M. Z. Alam, I. A. Yeasmin, and M. N. Sharif, “Analytical technique for solving Van der Pol oscillator,” *J. Eng. Appl. Sci.*, vol. 8, no. 1, pp. 29–32, 2024.
- [23] S. Shivaranjini and N. Srivastava, “Semi-analytical approach for solving the mathematical model of solid-phase diffusion in electrodes: An application of modified differential transform method,” *JPDE*, vol. 13, no. 1, Art. no. 101107, 2025.
- [24] N. H. Aljahdaly and M. A. Alharbi, “Semi-analytical solution of non-homogeneous Duffing oscillator equation by the Padé differential transformation algorithm,” *J. Low Freq. Noise Vib. Act. Control*, vol. 41, no. 4, pp. 1454–1465, 2022.
- [25] M. Huq, M. K. Hasan, and M. S. Alam, “An approximation technique for solving anti-symmetric constant force and anti-symmetric quadratic nonlinear oscillators,” *J. Low Freq. Noise Vib. Act. Control*, vol. 43, no. 4, pp. 1498–1508, 2024.