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STRUCTURAL PROPERTIES OF TOPOLOGICAL GRAPHS

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***Abstract**: This article explores the interface between topology and graph theory by associating a graph to each finite topological space and studying how topological properties are reflected in the corresponding graph. We prove that homeomorphic finite topological spaces yield isomorphic graphs. We also obtain a lower bound on the number of distinct non- topologies on an -point set.*

Keywords: *Topological graphs, Finite topological space, T_0 and T_1 spaces, Inequivalent topology, Distinct topology.*

AMS Subject classification: *05C10*

1 Introduction

Graph theory is deeply connected with many areas of mathematics. Beck [3] linked graph theory with ring theory, while Angsuman Das [4] connected it with linear algebra by defining the nonzero component graph of a finite-dimensional vector space. Numerous other algebraic structures have similarly been studied through a graph-theoretic lens. Well-known connections also exist between graph theory and combinatorics.

More recently, several works have related graph theory and topology. In 2022, Aniyan and Naduvath [1] defined topologies on graphs by considering collections of subgraphs that satisfy the axioms of point-set topology, and later examined subspace graph topologies [2]. In 2023, Lalithambigai and Gnanachandra [5] studied topological structures induced by the chromatic partition of the vertex set. Topologies induced by various graph metrics—including detour, geodesic, circular, and D-circular distances—were investigated in [6]. Maheswari and Durgadevi [7] introduced a micro topological space induced by a simple graph and analyzed micro-connectivity via graph theory. Nasef *et al.* [9] studied properties of nano topologies induced by graphs.

We recall standard notions from graph theory and elementary topology. A graph is an ordered pair $G = (V, E)$ where V is a nonempty set and $E \subseteq V \times V$. Elements of V are *vertices*, and elements of E are *edges*. Two vertices x and y are *adjacent* if there is an edge between them. A graph is *simple* if it has no loops or parallel edges. An *empty* (or *null*) graph has no edges. A graph is *complete* if every two distinct vertices are adjacent. A graph is *regular* if all vertices have the same degree. Graphs G_1 and G_2 are *isomorphic* if there is a bijection between their vertex sets that preserves adjacency. The join $G = G_1 + G_2$ of two graphs on disjoint vertex sets V_1 and V_2 with edge sets E_1 and E_2 is the union $G_1 \cup G_2$ together with all edges joining V_1 and V_2. Further terminology appears in [11].

Let X be a finite set and let T be a topology on X. A *finite topological space* is a topological space with a finite underlying set. The discrete topology is the finest topology on

X; here every subset is open. The indiscrete (or trivial) topology is the coarsest topology on X, where the only open sets are $\varnothing$ and X. A space X is T_0 (Kolmogorov) if, for every pair of distinct points, at least one has a neighbourhood not containing the other. The Sierpiński space is a two-point finite space in which exactly one point is closed; it is the smallest example that is neither discrete nor indiscrete and plays an important role in semantics via the Scott topology. Standard references include [8], and a classical guide to finite spaces is due to Stong [10].

Finite spaces have attracted renewed attention, in part because many applications in digital and image processing begin with finite data sets. In this context, T_0 spaces are particularly natural.

Lemma 1.1 *Let X be finite. If (X,T) is T_1, then it is discrete.*

Proof. Every singleton is closed in a T_1 space; in the finite setting, every set is a finite union of singletons and hence closed. Therefore every subset is also open, so the topology is discrete.

Lemma 1.2 [10] $T_2 \Rightarrow T_1 \Rightarrow T_0$.

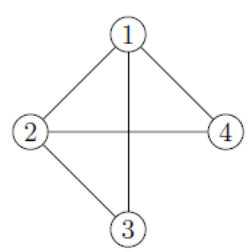
2 Topological graphs

Let (X,T) be a finite topological space. Define the *topological graph* G_T on the vertex set $V(G_T) = X$ by declaring two distinct vertices $x_1, x_2 \in X$ to be adjacent if and only if there exist distinct open sets $G_1, G_2 \in T$ with $x_1 \in G_1$ and $x_2 \in G_2$.

By definition, adjacency is symmetric, so G_T is a simple undirected graph. When T is fixed we write G_T for the associated topological graph.

Observations.

1. If (X,T) is indiscrete with $|X| = n$, then G_T is the null graph on n vertices.
2. If (X,T) is discrete with $|X| = n$, then G_T is the complete graph on n vertices.
3. If $X = \{1,2,3,4\}$ and $T = \{\varnothing, \{1\}, \{1,2\}, X\}$, then the corresponding graph is:



3 Properties of topological graphs

We record several basic properties linking topological and graph-theoretic structure.

Theorem 3.1 *If (X,T) is T_0, then G_T is complete.*

Proof. Let $x \neq y$ in X. Since T is T_0, there exists an open set G with $x \in G$ and $y \notin G$. As

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$G \neq X$, the sets G and X are distinct open sets containing x and y respectively. Hence x and y are adjacent, so G_T is complete.

The converse need not hold. For example, let $X = \{1,2,3\}$ and $T = \{\varnothing, \{1,2\}, X\}$. Then T is not T_0, but the associated graph is complete:

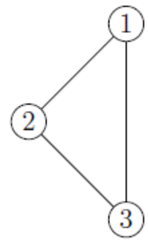

Figure 1

Corollary 1 *If (X,T) is T_1 or T_2, then G_T is complete.*

Proof. This follows from Theorem 3.1 and Lemma 1.2.

Theorem 3.2 *Let $(X,T) \subseteq (X,T')$ be topologies on the same set X (so T' is finer). Then G_T is a subgraph of $G_{T'}$.*

Proof. Suppose $x_1 \neq x_2$ are adjacent in G_T. Then there exist distinct open sets $G_1, G_2 \in T$ with $x_1 \in G_1$ and $x_2 \in G_2$. Since $T \subseteq T'$, the sets G_1, G_2 are open in T', so x_1 and x_2 are adjacent in $G_{T'}$ as well.

Theorem 3.3 *Let (X, T_X) and (Y, T_Y) be finite topological spaces. If X and Y are homeomorphic, then G_{T_X} and G_{T_Y} are isomorphic.*

Proof. Let $f: X \to Y$ be a homeomorphism. If $x_1 \neq x_2$ are adjacent in G_{T_X}, then there exist distinct open sets $G_1, G_2 \subseteq X$ with $x_1 \in G_1$ and $x_2 \in G_2$. Choose $x \in G_1 \backslash G_2$. Then $f(x) \in f(G_1)$ and $f(x) \notin f(G_2)$, so $f(G_1) \neq f(G_2)$. As f is a homeomorphism, $f(G_1)$ and $f(G_2)$ are open in Y, with $f(x_1) \in f(G_1)$ and $f(x_2) \in f(G_2)$. Hence $f(x_1)$ and $f(x_2)$ are adjacent in G_{T_Y}. A symmetric argument using f^{-1} completes the proof, so f induces a graph isomorphism $G_{T_X} \cong G_{T_Y}$.

The converse fails. For instance, with $X = \{1,2,3,4\}$, let $T_1 = \{\varnothing, \{1\}, \{1,2\}, X\}$ and $T_2 = \{\varnothing, \{3,4\}, X\}$. These topologies are not homeomorphic, yet their associated graphs are isomorphic (see Figure 3).

(a) Graph G_{T_1} (b) Graph G_{T_2}

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Figure 2: Two isomorphic graphs from non-homeomorphic topologies

Theorem 3.4 *Let $|X| \geq 2$ be finite and T a topology on X. Then G_T is connected if and only if T is not indiscrete.*

Proof. Suppose G_T is connected. Then there exist $x_1, x_2 \in X$ with x_1 adjacent to x_2, so there are open sets G_1, G_2 with $x_1 \in G_1$ and $x_2 \in G_2$. As G_1 and G_2 are nonempty and distinct, at least one (say G_1) satisfies $G_1 \neq X$. Thus T is not indiscrete.

Conversely, if T is not indiscrete, choose an open set G with $\varnothing \neq G \neq X$. Pick $x \in G$. For any $y \neq x$, the sets G and X are distinct open sets containing x and y, respectively; hence x is adjacent to every other vertex, and G_T is connected.

Corollary 2 *If $|X| = n$ and T is not indiscrete, then the domination number of G_T is* 1.

Proof. By Theorem 3.4, there is a vertex adjacent to every other vertex; this single vertex forms a dominating set.

Theorem 3.5 *If G_T is regular of order n, then G_T is either complete or null.*

Proof. Assume G_T is regular and not null. Then T is not indiscrete, so by Corollary 2 there exists a vertex of degree $n-1$. Regularity forces every vertex to have degree $n-1$, so G_T is complete. The converse is immediate.

Theorem 3.6 *Let (X, T_X) be a topological space and $Y \subseteq X$ with the subspace topology T_Y. If G_{T_X} and G_{T_Y} are the corresponding topological graphs, then G_{T_Y} is a subgraph of G_{T_X}.*

Proof. Let $y_1 \neq y_2$ in Y be adjacent in G_{T_Y}. There exist distinct open sets G_{1Y}, G_{2Y} in Y with $y_1 \in G_{1Y}$ and $y_2 \in G_{2Y}$. Since G_{1Y} and G_{2Y} are open in the subspace topology, there are open sets G_1, G_2 in X such that $G_{1Y} = G_1 \cap Y$ and $G_{2Y} = G_2 \cap Y$. Choose $y \in G_{1Y} \setminus G_{2Y}$. Then $y \in G_1$ and $y \notin G_2$, so $G_1 \neq G_2$ and $y_1 \in G_1$, $y_2 \in G_2$. Hence y_1 and y_2 are adjacent in G_{T_X}.

Definition 1 ***Lexicographic product of graphs.*** *The lexicographic product (graph composition) $G \cdot H$ has vertex set $V(G) \times V(H)$; two vertices (u, v) and (x, y) are adjacent in $G \cdot H$ if and only if either u is adjacent to x in G, or $u = x$ and v is adjacent to y in H.*

Definition 2 ***Simple product of graphs.*** *The simple product $G *_s H$ has vertex set $V(G) \times V(H)$; two vertices (u, v) and (x, y) are adjacent in $G *_s H$ if and only if u is adjacent to x in G or v is adjacent to y in H.*

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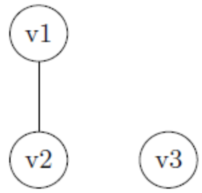
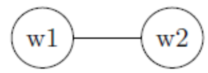


(a) G



(b) H

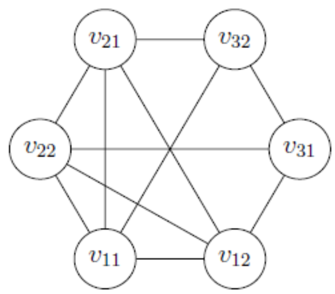


Figure 3 Simple product of G and H (where $v_{ij} = v_i * w_j$)

Theorem 3.7 *Let* X *and* Y *be finite topological spaces with associated graphs* G_X *and* G_Y. *Then both the lexicographic product* $G_X \cdot G_Y$ *and the simple product* $G_X *_s G_Y$ *are subgraphs of the graph* $G_{X\times Y}$ *associated to the product topology on* $X \times Y$.

Proof. The vertex sets of $G_X \cdot G_Y$, $G_X *_s G_Y$, and $G_{X\times Y}$ are all $X \times Y$.

First, suppose (x_1, y_1) is adjacent to (x_2, y_2) in $G_X \cdot G_Y$. Then either x_1 is adjacent to x_2 in G_X, in which case there exist distinct open sets $G_{X1}, G_{X2} \subseteq X$ with $x_1 \in G_{X1}$ and $x_2 \in G_{X2}$, so $(x_1, y_1) \in G_{X1} \times Y$ and $(x_2, y_2) \in G_{X2} \times Y$, which are distinct open sets in $X \times Y$; or $x_1 = x_2$ and y_1 is adjacent to y_2 in G_Y, in which case there are distinct open sets $G_{Y1}, G_{Y2} \subseteq Y$ with $y_1 \in G_{Y1}$ and $y_2 \in G_{Y2}$, so $(x_1, y_1) \in X \times G_{Y1}$ and $(x_2, y_2) \in X \times G_{Y2}$. In either case, (x_1, y_1) and (x_2, y_2) are adjacent in $G_{X\times Y}$. Thus $G_X \cdot G_Y$ is a subgraph of $G_{X\times Y}$.

For the simple product, if (x_1, y_1) is adjacent to (x_2, y_2) in $G_X *_s G_Y$, then either x_1 is adjacent to x_2 in G_X or y_1 is adjacent to y_2 in G_Y. The same arguments as above produce distinct open sets in $X \times Y$ containing (x_1, y_1) and (x_2, y_2), so $G_X *_s G_Y$ is also a subgraph of $G_{X\times Y}$.

Remarks.

1. In the previous theorem, the lexicographic product may be a proper subgraph of $G_{X\times Y}$; see the example below.
2. Likewise, the simple product may be a proper subgraph of $G_{X\times Y}$.

For instance, take $X = \{a, b, c\}$ with $T_X = \{\varnothing, \{a\}, X\}$ and $Y = \{x, y\}$ with $T_Y = \{\varnothing, Y\}$. If G_X and G_Y are the associated graphs, then both $G_X \cdot G_Y$ and $G_X *_s G_Y$ are strict spanning subgraphs of $G_{X\times Y}$.

4 Counting topological graphs

A finite graph G is a *topological graph* if there exists a finite topological space (X, T)

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such that the graph associated to T is isomorphic to G. For example, the graph

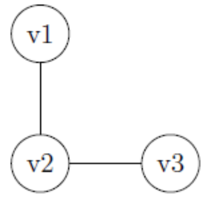

Figure 4

is topological because for $X = \{a, b, c\}$ with $T = \{\varnothing, \{a\}, X\}$ the associated topological graph is isomorphic to G.

Lemma 4.1 *The number of topological graphs of order n is at least n.*

Proof. Consider the graphs $\overline{K_n}$, $K_1 + \overline{K_{n-1}}$, $K_2 + \overline{K_{n-2}}$, ... , $K_{n-1} + \overline{K_1}$, K_n. On the set $X = \{x_1, \dots, x_n\}$, take the indiscrete topology, then the topologies

$$T_1 = \{\varnothing, \{x_1\}, X\}, \quad T_2 = \{\varnothing, \{x_1, x_2\}, X\}, \quad \dots, \quad T_{n-1} = \{\varnothing, \{x_1, \dots, x_{n-1}\}, X\},$$

and finally the discrete topology. These correspond respectively to the graphs above. Hence there are at least n distinct topological graphs of order n.

Theorem 4.2 *If G is a topological graph and x a vertex of G, then $G - \{x\}$ is also a topological graph.*

Proof. Let G be associated to (X, T), where $X = \{x_1, \dots, x_n\}$. Fix $x \in X$ and set $Y = X \backslash \{x\}$. Define $T_Y = \{ G_i \backslash \{x\} | G_i \in T \}$.
Claim (i). T_Y is a topology on Y. Indeed, $\varnothing \in T$ implies $\varnothing \in T_Y$, and $X \in T$ implies $Y = X \backslash \{x\} \in T_Y$. If $G_1 \backslash \{x\}, G_2 \backslash \{x\} \in T_Y$, then

$$(G_1 \backslash \{x\}) \cup (G_2 \backslash \{x\}) = (G_1 \cup G_2) \backslash \{x\} \in T_Y,$$

and similarly for finite intersections. Thus T_Y is a topology on Y.
Claim (ii). The graph associated to (Y, T_Y) is $G - \{x\}$. Indeed, the vertex set is Y, and if $x_i \neq x_j$ with $1 \leq i, j \leq n - 1$ are adjacent in G, there exist distinct open sets $G_i, G_j \in T$ with $x_i \in G_i$ and $x_j \in G_j$. Then $x_i \in G_i \backslash \{x\}$ and $x_j \in G_j \backslash \{x\}$, which are distinct open sets in T_Y, so x_i and x_j are adjacent in the associated graph. Hence $G - \{x\}$ is a topological graph.

Remark. If G is a topological graph and e is an edge of G, the graph $G - e$ need not be a topological graph. For example:

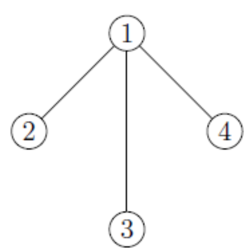

Figure 5: A graph G for which $G - e$ is not necessarily topological

Theorem 4.3 *Let $|X| = n$. Then the number of non-equivalent T_0 finite topological spaces on X is at least $n - 1$.*

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Proof. By Lemma 4.1 there are at least $n-1$ noncomplete topological graphs of order n. Applying Theorem 4.2 yields the claim.

Theorem 4.4 *The number of distinct non-T_0 topologies on a set X with $|X|=n$ is at least*

$$1+n+n(n-1)+n(n-1)(n-2)+\cdots+n(n-1)\cdots 4\cdot 3.$$

Proof. By Lemma 4.1 there are at least n topological graphs of order n, of which $n-1$ are noncomplete. By Theorem 4.2 we obtain corresponding non-T_0 topologies. Taking permutations of the vertices gives the stated lower bound.

5 Conclusion

We investigated finite topological spaces via their associated graphs. Since many data-analytic settings begin with finite data, such a translation enables the use of graph-theoretic tools for studying finite topologies and related structures. Several further directions are natural: the study of other graph products in this framework, sharper bounds for the enumeration of finite topologies, and structural parameters (such as domination and connectivity) for the associated topological graphs.

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