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COLORING AND GRAPH INVARIANTS ANALYSIS OF CUPOLA INSPIRED GRAPHS

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Abstract: *Graph coloring and topological invariants are fascinating subjects, particularly when applied to physical structures in engineering and architecture. These concepts help to analyze the characteristics of structures by representing them as nodes and arcs. Graph coloring involves applying minimal colors to a graph in such a way that no two adjacent nodes share the same color. In this paper, we focus on applying graph coloring to novel cupola-like structures, specifically defined as the conical graph (monolithic domes) and the novel Brunelleschi graph (Brunelleschi dome). Further generate the computation method for finding four important degree-based graphs invariants of these cupola graphs.*

Keywords: Cupola structures, Graph coloring, Temperature indices, Randić index, Sombor index

1. Introduction

Graph theory is a branch of mathematics that studies the relationships between the subjected sources through structures known as graphs. For this we design a graph by means of nodes set and connecting them by means of arcs set denoted by $V(G)$ and $E(G)$ respectively. The number of arc incident with any node v_i of a graph G is said to be its *degree* [4], denoted by $\delta(v_i)$.

A cupola structure is a geometric form featuring a dome-like or pyramidal shape. In architecture, cupolas are often used as a self-reinforced and super insulated towers. They belong to the family of self-supporting which form a arch. In mathematical and structural studies, cupola-like forms are analyzed for their symmetry and connectivity properties. These structures also find applications in graph theory, chemistry, and material science, where their unique configurations contribute to stability and efficiency in various designs.

Here, we define a graph $G(m,n)$, a novel structure where a complete graph K_n is positioned at the apex. This apex is expanded by attaching m cycles (C_n), each having n vertices, resulting in a Brunelleschi dome-like formation [2, 6, 8]. The graph consists of $|V(G)| = n(m+1)$ vertices and $|E(G)| = \frac{n^2 + 4mn - n}{2}$ edges, containing cycles $C_n^1, C_n^2, C_n^3, \ldots, C_n^m$ as shown in the **Figure 1**. This structure is named the *Brunelleschi graph* $G(m,n)$. Where $n = 3, 4, 5, \ldots$ represents the number of vertices in each level, and m indicates the number of levels.

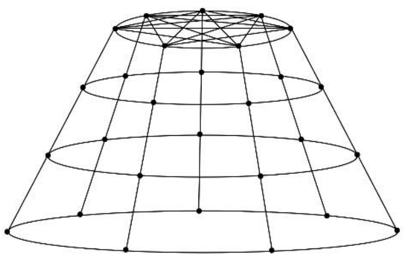

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Figure 1. Brunelleschi graph *G (3, 7)*

In 2020, Ayache et al., [1] introduced the novel graph called conical graph $G(m,n)$ resembles monolithic dome like structure [5], consists of $nm+1$ nodes and $2nm$ arcs, which contains a apex(center O) and m number of cycles with n number of nodes ($C_n^1, C_n^2, C_n^3, ..., C_n^m$), see **Figure 2**.

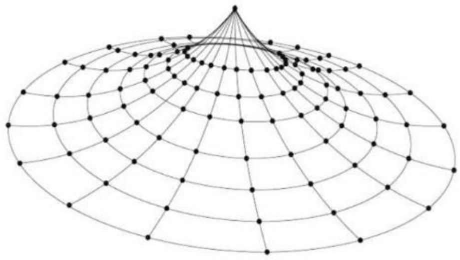

Figure 2. Conical graph *G (7, 19)*

Graph coloring is a fundamental concept in graph theory where colors are assigned to nodes, arcs, or regions of a graph following specific rules. The most common type is node(vertex) coloring, where adjacent nodes must have different colors. The graph coloring in means of architecture gives the approximation of the strength the physical structure acquires and adds some base to the idea so as to find the failure of construction leading to their destruction by referring the minimum number of colors required and their respective assignment in the structure. This idea proposes the better understanding of the independence of the nodes from each other.

In this work, dome shaped frameworks are represented using two graph models, namely, the conical graph [1] and the novel Brunelleschi graph. Each vertex in these graphs denotes a structural joint, and each edge represents a connecting member of the dome. The apex arrangement plays an important role in structural behavior. In the Brunelleschi model, the apex forms a complete graph that distributes load uniformly, while in the conical model, a single apex vertex transmits the entire load along radial connections. Applying graph coloring to these structures enables the identification of independent regions of load transmission, ensuring that no two adjacent joints belong to the same stress zone. Such coloring helps in separating force paths, minimizing local stress conflicts, and improving overall balance and resistance to failure.

Topological indices are numerical values assigned to a graph based on its topology. These indices find applications in chemistry, particularly in QSPR and QSAR studies, where researchers seek to correlate molecular structure with biological activity or other properties. See [10, 11] to study the application of distance-based status-connectivity indices in QSPR and QSAR studies. In this paper, we have considered characteristic study by means of coloring and computing topological invariants namely Randić index, Sombor index, First temperature index, Second temperature index to provide further insight into stability characteristics of these two dome structures.

2. Preliminaries

Graph coloring [13] is the practice of assigning colors to a graph's components called nodes or arcs, so that no two adjacent elements belong to same color class.

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The *chromatic number* [13] of a graph is the minimum number of color class required to assign the nodes so that adjacent nodes belong to two different color class. It is typically represented as $\chi(G)$ for a given graph *G*.

The topological indices we will compute here are:

- First temperature index [9]:
  $$T_1(G) = \sum_{uv \in E(G)} \left[T(u) + T(v)\right] \quad (1)$$
- Second temperature index [7]:
  $$T_2(G) = \sum_{uv \in E(G)} \left[T(u) \times T(v)\right] \quad (2)$$
  Where, $T(u) = \dfrac{\delta(u)}{|V(G)| - \delta(u)} \quad \forall \ u \in V(G)$.
- Randić index [12]:
  $$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{\delta(u)\delta(v)}} \quad (3)$$
- Sombor index [3]:
  $$SO(G) = \sum_{uv \in E(G)} \sqrt{\delta^2(u) + \delta^2(v)} \quad (4)$$

3. Coloring of cupola inspired conical and Brunelleschi graphs *G(m, n)*

Theorem 3.1 *The chromatic number of any conical graph* $\chi(G(m,n))$ *with* $n \geq 4$ *is always 3 whenever n is even and 4 whenever n is odd.*

Proof: According to the definition, conical graph is nothing but the set of *m* cycles with *n* number of vertices and each vertex in level *i* is connected to corresponding vertex in level *i+1*, for $i = 1,2,3,\ldots,m-1$ and includes singleton vertex at apex point which is connected to all *n* vertices in the cycle at level *m*=1, which form a wheel $W_{1,n}$.

By the definition of chromatic number $\chi(G)$ we need to find minimum number of colors required to provide a proper coloring to conical graph. Therefore,

$$\chi(G(m,n)) = \max\left\{\chi(W_{1,n}),\ \chi(C_n)\right\}$$
$$= \max\left\{\begin{matrix} 3, & 2 \ \ if \ \ n \ is \ even \\ 4, & 3 \ \ if \ \ n \ is \ odd \end{matrix}\right\}$$
$$\chi(G(m,n)) = \left\{\begin{matrix} 3 \ \ if \ \ n \ is \ even \\ 4 \ \ if \ \ n \ is \ odd \end{matrix}\right\}$$

Example 3.1: For the conical graph $G(8,6)$, the chromatic number is found to be 3. Similarly, for conical graph $G(6,9)$, the chromatic number is 4 as shown in the **Figure 3** and **Figure 4.**

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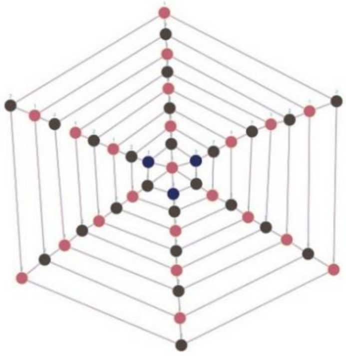

Figure 3. Coloring of conical graph *G (8,6)*

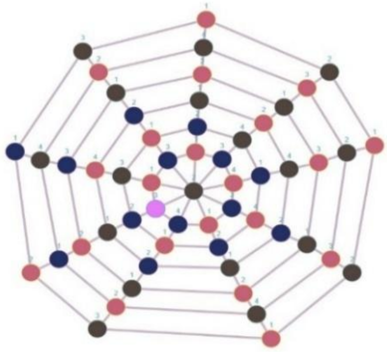

Figure 4. Coloring of conical graph *G (6,9)*

Theorem 3.2 *The chromatic number of any Brunelleschi graph* $\chi(G(m,n))$ *with* $n \geq 3$ *is always n.*

Proof: According to the definition, Brunelleschi graph is nothing but the set of *m* cycles with *n* number of vertices and each vertex in level *i* is connected to corresponding vertex in level *i+1*, for $i = 1,2,3,...,m-1$ and includes a complete graph K_n at apex which is connected to all *n* vertices in the cycle at level *m=1* by an edge between its corresponding vertices.

By the definition of chromatic number $\chi(G)$ we need to find minimum number of colors required to provide a proper coloring to Brunelleschi graph. Therefore,

$$\chi(G(m,n)) = \max\{\chi(K_n), \chi(C_n)\}$$

$$= \max\begin{Bmatrix} n, & 2 & if \ n \ is \ even \\ n, & 3 & if \ n \ is \ odd \end{Bmatrix}$$

$$\chi(G(m,n)) = \begin{Bmatrix} n & if \ n \ is \ even \\ n & if \ n \ is \ odd \end{Bmatrix}$$

Example 3.2: For the Brunelleschi graph $G(4,3)$, the chromatic number is found to be 3. Similarly, for Brunelleschi graph $G(5,6)$, the chromatic number is 4 as shown in the **Figure 5** and **Figure 6.**

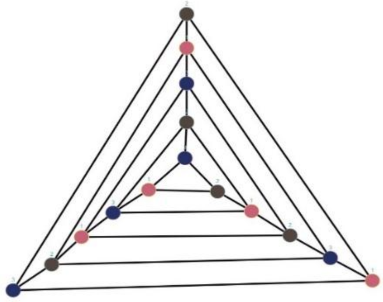

Figure 5. Coloring of Brunelleschi graph *G (4,3)*

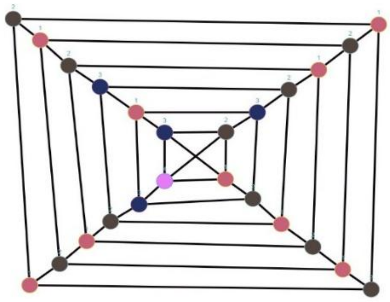

Figure 6. Coloring of Brunelleschi graph *G (5,6)*

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4. Computation of graph invariants

In this section, we compute the different steps involved in finding the above mentioned graph invariants for conical graph and Brunelleschi graph. These steps are basically dependent on node set and arc set partitions and their numerical values.

Theorem 4.1 *Consider* $G(m,n)$ *as a conical graph with m number of levels and n number of nodes in each level,* $n \geq 4$ *and* $|V(G)| = nm+1$. *Then,*

$$T_1(G(m,n)) = \frac{n^2}{n(m-1)+1} + \frac{16n(m-1)}{mn-3} + \frac{9n}{mn-2}$$

$$T_2(G(m,n)) = \frac{2n}{mn-3}\left(\frac{2n}{n(m-1)+1} + \frac{6}{mn-2} + \frac{8(2m-3)}{mn-3}\right) + \frac{9n}{(mn-2)^2}$$

$$R(G(m,n)) = n\left(\frac{12\sqrt{3}-30}{72}\right) + \frac{2mn}{4} + \frac{\sqrt{n}}{2}$$

$$SO(G(m,n)) = n\left(\sqrt{16+n^2} + 8\sqrt{2}m - 9\sqrt{2} + 5\right)$$

Proof: The node set of $G(m,n)$ can be written as:

$$V(G(m,n)) = \left\{\{O\},\{v_1^1, v_2^1, v_3^1,\ldots,v_n^1\},\{v_1^2, v_2^2, v_3^2,\ldots,v_n^2\},\ldots,\{v_1^{m-1}, v_2^{m-1}, v_3^{m-1},\ldots,v_n^{m-1}\},\{v_1^m, v_2^m, v_3^m,\ldots,v_n^m\}\right\}$$

These nodes can be partitioned into three different classes based on their respective degree.

$$\text{Degree of each node} = \begin{cases} n & : \quad v = O \\ 3 & : \quad v = v_i^m,\ i = 1,2,3,\ldots,n \\ 4 & : \quad v = Otherwise \end{cases}$$

The arc set of $G(m,n)$ can be written as,

$$E(G) = \bigcup_{k=0}^{3} E_k : \bigcap E_k = \phi .$$

This arc set can be partitioned into four different classes based on their respective degree of nodes which joined by them.

$E_0 = \left\{Ov_1^1, Ov_2^1, \ldots, Ov_n^1\right\}$ where, $|E_0| = n$.

$E_1 = \left\{v_1^{m-1}v_1^m, v_2^{m-1}v_2^m, \ldots, v_n^{m-1}v_n^m\right\}$ where, $|E_1| = n$.

And $E_3 = E_+ \cup E_*$. Such that $\forall\, j = 1,2,3,\ldots,m-1$, E_+ and E_* can be written as:

$E_+ = \left\{v_1^j v_2^j, v_2^j v_3^j, v_3^j v_4^j, \ldots, v_{n-1}^j v_n^j, v_n^j v_1^j\right\}$ where, $|E_+| = n(m-1)$.

$E_* = \left\{v_1^{j-1}v_1^m, v_2^{j-1}v_2^m, v_3^{j-1}v_3^m, \ldots, v_n^{j-1}v_n^m\right\}$ where, $|E_*| = n(m-2)$

Thus, $|E_3| = |E_+| + |E_*| = n(m-1) + n(m-2) = n(2m-3)$

We study these arc sets under four cases depending on their node degree and temperature.

Case 1: if $uv \in E_0$, then $\delta(u) = n,\ \delta(v) = 4$ and $T(u) = \dfrac{n}{n(m-1)+1}$, $T(v) = \dfrac{4}{mn-3}$.

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Case 2: if $uv \in E_1$, then $\delta(u)=4,\ \delta(v)=3$ and $T(u)=\frac{4}{mn-3},\ T(v)=\frac{3}{mn-2}$.

Case 3: if $uv \in E_2$, then $\delta(u)=3,\ \delta(v)=3$ and $T(u)=\frac{3}{mn-2},\ T(v)=\frac{3}{mn-2}$.

Case 4: if $uv \in E_3$, then $\delta(u)=4,\ \delta(v)=4$ and $T(u)=\frac{4}{mn-3},\ T(v)=\frac{4}{mn-3}$.

Therefore,

$$\begin{aligned} T_1(G(m,n)) &= \sum_{uv\in E_0}\left(\frac{n}{n(m-1)+1}+\frac{4}{mn-3}\right)+\sum_{uv\in E_1}\left(\frac{4}{mn-3}+\frac{3}{mn-2}\right) \\ &\quad+\sum_{uv\in E_2}\left(\frac{3}{mn-2}+\frac{3}{mn-2}\right)+\sum_{uv\in E_3}\left(\frac{4}{mn-3}+\frac{4}{mn-3}\right) \\ &= n\left(\frac{n}{n(m-1)+1}+\frac{4}{mn-3}\right)+n\left(\frac{4}{mn-3}+\frac{3}{mn-2}\right) \\ &\quad+n\left(\frac{3}{mn-2}+\frac{3}{mn-2}\right)+n\left(2m-3\right)\left(\frac{4}{mn-3}+\frac{4}{mn-3}\right) \end{aligned}$$

$$T_1(G(m,n))=\frac{n^2}{n(m-1)+1}+\frac{16n(m-1)}{mn-3}+\frac{9n}{mn-2}$$

And

$$\begin{aligned} T_2(G(m,n)) &= \sum_{uv\in E_0}\left(\frac{n}{n(m-1)+1}\times\frac{4}{mn-3}\right)+\sum_{uv\in E_1}\left(\frac{4}{mn-3}\times\frac{3}{mn-2}\right) \\ &\quad+\sum_{uv\in E_2}\left(\frac{3}{mn-2}\times\frac{3}{mn-2}\right)+\sum_{uv\in E_3}\left(\frac{4}{mn-3}\times\frac{4}{mn-3}\right) \\ &= n\left(\frac{n}{n(m-1)+1}\times\frac{4}{mn-3}\right)+n\left(\frac{4}{mn-3}\times\frac{3}{mn-2}\right) \\ &\quad+n\left(\frac{3}{mn-2}\times\frac{3}{mn-2}\right)+n\left(2m-3\right)\left(\frac{4}{mn-3}\times\frac{4}{mn-3}\right) \end{aligned}$$

$$T_1(G(m,n))=\frac{2n}{mn-3}\left(\frac{2n}{n(m-1)+1}+\frac{6}{mn-2}+\frac{8(2m-3)}{mn-3}\right)+\frac{9n}{(mn-2)^2}$$

And

$$\begin{aligned} R(G(m,n)) &= \sum_{uv\in E_0}\frac{1}{\sqrt{(n)(4)}}+\sum_{uv\in E_1}\frac{1}{\sqrt{(4)(3)}}+\sum_{uv\in E_2}\frac{1}{\sqrt{(3)(3)}}+\sum_{uv\in E_3}\frac{1}{\sqrt{(4)(4)}} \\ &= n\left(\frac{1}{\sqrt{4n}}\right)+n\left(\frac{1}{\sqrt{12}}\right)+n\left(\frac{1}{\sqrt{9}}\right)+n(2m-3)\left(\frac{1}{\sqrt{16}}\right) \end{aligned}$$

$$R(G(m,n))=n\left(\frac{12\sqrt{3}-30}{72}\right)+\frac{2mn}{4}+\frac{\sqrt{n}}{2}$$

And

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$$SO(G(m,n)) = \sum_{uv\in E_0}\sqrt{n^2+4^2} + \sum_{uv\in E_1}\sqrt{4^2+3^2} + \sum_{uv\in E_2}\sqrt{3^2+3^2} + \sum_{uv\in E_3}\sqrt{4^2+4^2}$$
$$= n\left(\sqrt{16+n^2}\right) + n\left(\sqrt{25}\right) + n\left(\sqrt{18}\right) + n(2m-3)\left(\sqrt{32}\right)$$
$$SO(G(m,n)) = n\left(\sqrt{16+n^2} + 8\sqrt{2}m - 9\sqrt{2} + 5\right)$$

Theorem 4.2 *Consider* $G(m,n)$ *as a Brunelleschi graph with m number of levels and n number of nodes in each level,* $n \geq 3$ *and* $|V(G)| = mn + n = n(m+1)$. *Then,*

$$T_1(G(m,n)) = \frac{n^2}{m} + \frac{16n(m-1)}{n(m+1)-4} + \frac{9n}{n(m+1)-3}$$
$$T_2(G(m,n)) = \frac{n(n-1)}{2m^2} + \frac{4n}{m(n(m+1)-4)} + \frac{12n}{(n(m+1)-4)(n(m+1)-3)} + \frac{9n}{(n(m+1)-3)^2} + \frac{16n(2m-3)}{(n(m+1)-4)^2}$$
$$R(G(m,n)) = \frac{6mn+n-6+6\sqrt{n}}{12} + \frac{n}{2\sqrt{3}}$$
$$SO(G(m,n)) = \frac{\sqrt{2}n^3 - \sqrt{2}n^2}{2} + n\sqrt{16+n^2} + (5-9\sqrt{2})n + 8\sqrt{2}mn$$

Proof: The node set of $G(m,n)$ can be written as:

$$V(G(m,n)) = \left\{\{u_1^0, u_2^0, u_3^0, ..., u_n^0\}, \{u_1^1, u_2^1, u_3^1, ..., u_n^1\}, \{u_1^2, u_2^2, u_3^2, ..., u_n^2\}, ..., \{u_1^m, u_2^m, u_3^m, ..., u_n^m\}\right\}$$

These nodes can be partitioned into three different classes based on their respective degree.

$$\text{Degree of each node} = \begin{cases} n & : \quad u = u_i^0, \ \ i = 1,2,3,...,n \\ 3 & : \quad u = u_i^m \\ 4 & : \quad u = Otherwise \end{cases}$$

The arc set of $G(m,n)$ can be written as,

$$E(G) = \bigcup_{k=0}^{4} E_k : \bigcap E_k = \phi.$$

This arc set can be partitioned into five different classes based on their respective degree of nodes which joined by them.

$$E_0 = \{\{u_1^0u_2^0, u_2^0u_3^0, ..., u_{n-1}^0u_n^0, u_n^0u_1^0\}, \{u_1^1u_3^1, u_1^1u_4^1, u_1^1u_5^1..., u_1^1u_{n-1}^1\}, \{u_2^1u_4^1, u_2^1u_5^1, u_2^1u_6^1..., u_2^1u_n^1\},$$
$$\{u_3^1u_1^1, u_3^1u_5^1, u_3^1u_6^1..., u_3^1u_n^1\}, ..., \{u_n^1u_2^1, u_n^1u_3^1, u_n^1u_4^1..., u_n^1u_{n-2}^1\}\} \Rightarrow |E_0| = \frac{n(n-1)}{2}.$$

$$E_1 = \left\{u_1^0u_1^1, u_2^0u_2^1, ..., u_n^0u_n^1\right\} \Rightarrow |E_1| = n.$$

$$E_2 = \left\{u_1^{m-1}u_1^m, u_2^{m-1}u_2^m, ..., u_n^{m-1}u_n^m\right\} \Rightarrow |E_2| = n.$$

$$E_3 = \left\{u_1^mu_2^m, u_2^mu_3^m, ..., u_{n-1}^mu_n^m, u_n^mu_1^m\right\} \Rightarrow |E_3| = n.$$

And $E_4 = E_+ \bigcup E_*$. Such that $\forall\, j = 1,2,3,...,m-1$, E_+ and E_* can be written as:

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$E_+ = \{u_1^j u_2^j, u_2^j u_3^j, u_3^j u_4^j, \ldots, u_{n-1}^j u_n^j, u_n^j u_1^j\} \Rightarrow |E_+| = n(m-1)$.

$E_* = \{u_1^{j-1} u_1^m, u_2^{j-1} u_2^m, u_3^{j-1} u_3^m, \ldots, u_n^{j-1} u_n^m\} \Rightarrow |E_*| = n(m-2)$

Thus, $|E_4| = |E_+| + |E_*| = n(m-1) + n(m-2) = n(2m-3)$.

We study these arc sets under five cases depending on their node degree and temperature.

Case 1: if $uv \in E_0$, then $\delta(u) = \delta(v) = n$ and $T(u) = T(v) = \frac{1}{m}$.

Case 2: if $uv \in E_1$, then $\delta(u) = n,\ \delta(v) = 4$ and $T(u) = \frac{1}{m},\ T(v) = \frac{4}{n(m+1)-4}$.

Case 3: if $uv \in E_2$, then $\delta(u) = 4,\ \delta(v) = 3$, and $T(u) = \frac{4}{n(m+1)-4}, T(v) = \frac{3}{n(m+1)-3}$

Case 4: if $uv \in E_3$, then $\delta(u) = \delta(v) = 3$ and $T(u) = T(v) = \frac{3}{n(m+1)-3}$.

Case 5: if $uv \in E_4$, then $\delta(u) = \delta(v) = 4$ and $T(u) = T(v) = \frac{4}{n(m+1)-4}$.

Therefore,

$$T_1(G(m,n)) = \sum_{uv \in E_0} \left(\frac{1}{m} + \frac{1}{m} \right) + \sum_{uv \in E_1} \left(\frac{1}{m} + \frac{4}{n(m+1)-4} \right) + \sum_{uv \in E_2} \left(\frac{4}{n(m+1)-4} + \frac{3}{n(m+1)-3} \right)$$
$$+ \sum_{uv \in E_3} \left(\frac{3}{n(m+1)-3} + \frac{3}{n(m+1)-3} \right) + \sum_{uv \in E_4} \left(\frac{4}{n(m+1)-4} + \frac{4}{n(m+1)-4} \right)$$
$$= \frac{n(n-1)}{2} \left(\frac{1}{m} + \frac{1}{m} \right) + n \left(\frac{1}{m} + \frac{4}{n(m+1)-4} \right) + n \left(\frac{4}{n(m+1)-4} + \frac{3}{n(m+1)-3} \right)$$
$$+ n \left(\frac{3}{n(m+1)-3} + \frac{3}{n(m+1)-3} \right) + n(2m-3) \left(\frac{4}{n(m+1)-4} + \frac{4}{n(m+1)-4} \right)$$

$$T_1(G(m,n)) = \frac{n^2}{m} + \frac{16n(m-1)}{n(m+1)-4} + \frac{9n}{n(m+1)-3}$$

And

$$T_2(G(m,n)) = \sum_{uv \in E_0} \left(\frac{1}{m} \times \frac{1}{m} \right) + \sum_{uv \in E_1} \left(\frac{1}{m} \times \frac{4}{n(m+1)-4} \right) + \sum_{uv \in E_2} \left(\frac{4}{n(m+1)-4} \times \frac{3}{n(m+1)-3} \right)$$
$$+ \sum_{uv \in E_3} \left(\frac{3}{n(m+1)-3} \times \frac{3}{n(m+1)-3} \right) + \sum_{uv \in E_4} \left(\frac{4}{n(m+1)-4} \times \frac{4}{n(m+1)-4} \right)$$
$$= \frac{n(n-1)}{2} \left(\frac{1}{m} \times \frac{1}{m} \right) + n \left(\frac{1}{m} \times \frac{4}{n(m+1)-4} \right) + n \left(\frac{4}{n(m+1)-4} \times \frac{3}{n(m+1)-3} \right)$$
$$+ n \left(\frac{3}{n(m+1)-3} \times \frac{3}{n(m+1)-3} \right) + n(2m-3) \left(\frac{4}{n(m+1)-4} \times \frac{4}{n(m+1)-4} \right)$$

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$$T_2(G(m,n)) = \frac{n(n-1)}{2m^2} + \frac{4n}{m(n(m+1)-4)} + \frac{12n}{(n(m+1)-4)(n(m+1)-3)} + \frac{9n}{(n(m+1)-3)^2} + \frac{16n(2m-3)}{(n(m+1)-4)^2}$$

And

$$R(G(m,n)) = \sum_{uv\in E_0}\left(\frac{1}{\sqrt{n^2}}\right) + \sum_{uv\in E_1}\left(\frac{1}{\sqrt{4n}}\right) + \sum_{uv\in E_2}\left(\frac{1}{\sqrt{12}}\right) + \sum_{uv\in E_3}\left(\frac{1}{\sqrt{9}}\right) + \sum_{uv\in E_4}\left(\frac{1}{\sqrt{16}}\right)$$

$$= \frac{n(n-1)}{2}\left(\frac{1}{n}\right) + n\left(\frac{1}{2\sqrt{n}}\right) + n\left(\frac{1}{2\sqrt{3}}\right) + n\left(\frac{1}{3}\right) + n(2m-3)\left(\frac{1}{4}\right)$$

$$R(G(m,n)) = \frac{6mn+n-6+6\sqrt{n}}{12} + \frac{n}{2\sqrt{3}}$$

And

$$SO(G(m,n)) = \sum_{uv\in E_0}\left(\sqrt{2n^2}\right) + \sum_{uv\in E_1}\left(\sqrt{16+n^2}\right) + \sum_{uv\in E_2}\left(\sqrt{25}\right) + \sum_{uv\in E_3}\left(\sqrt{18}\right) + \sum_{uv\in E_4}\left(\sqrt{32}\right)$$

$$= \frac{n(n-1)}{2}\left(\sqrt{2}n\right) + n\left(\sqrt{16+n^2}\right) + n\left(5\right) + n\left(3\sqrt{2}\right) + n(2m-3)\left(4\sqrt{2}\right)$$

$$SO(G(m,n)) = \frac{\sqrt{2}n^3-\sqrt{2}n^2}{2} + n\sqrt{16+n^2} + (5-9\sqrt{2})n + 8\sqrt{2}mn$$

7. Conclusion

The graphs which we studied in this paper are defined as conical graph and Brunelleschi graph with *m* levels and *n* nodes in each level. These kind of graph structures come with various application to the field of engineering and helpful to study stability, data hierarchies and their visualization through graph coloring and topological indices. Coloring helps separate stress zones and prevent overlap. In the conical graph, the apex has a much higher degree than base vertices while in the Brunelleschi graph, the apex forms a complete graph, distributing the load among multiple top-level vertices leads to the variation in the computed indices which reflects the internal balance of the dome structures. In graph-based models, each vertex corresponds to a joint and its degree represents the number of connected members that share the load. When the degree distribution is nearly uniform, the overall stress across the structure becomes evenly balanced. Consequently, indices such as First temperature index, Second temperature index and Sombor index, which depend on differences in vertex degrees, attain smaller value for the Brunelleschi graph than for the conical graph. These lower values indicate that the load is shared more evenly among all joints in the Brunelleschi graph, thereby reducing localized stress and minimizing the possibility of structural failure, since its apex distributes load among multiple vertices unlike in the conical graph, where the apex vertex bears the entire load and higher degree. The results show that the Brunelleschi graph model offers greater stability and uniform load distribution compared to the conical form. Therefore, the combined use of graph coloring and topological indices provides a mathematical framework for predicting, comparing, and enhancing the stability of dome constructions.

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