ISSN: 1672 - 6553

JOURNAL OF DYNAMICS AND CONTROL

VOLUME 9 ISSUE 8: P232-241

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JOURNAL OF DYNAMICS AND CONTROL
DOI: https://doi.org/10.71058/jodac.v9i8016

ON BIPARTITE DOMINATION TOPOLOGICAL INDICES OF SOME GRAPHS

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Abstract: *For a nontrivial connected graph G, a nonempty set S⊆V (G) is a bipartite dominating set of graph G, if the subgraph G[S] induced by S is bipartite and for every vertex not in S is dominated by any vertex in S. In this paper, we define a new bipartite dominating degree (bdd) of each vertex x∈V(G), called bipartite domination degree of x and denoted by* $d_{b\gamma}(G)$. *Also, we establish exact value for the bdd zagreb indices of some class of graph.*
Keywords*: Domination, Bipartite dominating set, Bipartite domination number, Topological indices, Bipartite domination topological indices.*
AMS subject classification*:05C69, 05C76, 68R10.*

1. Introduction

Let $G = (V, E)$ be a simple undirected graph and let $n = |V(G)|$ and $m = |E(G)|$ be the order and size of the graph respectively. The degree of a vertex x in G is the number of edges that are incident to that in G and it is denoted by $d(x)$.

A non empty subset S of vertices in a graph G is a dominating set if every vertex u in $V-S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ is the smallest cardinality of a minimal dominating set of G. One of the fast developing fields in graph theory is the study of domination and related topics.

A non empty set $S \subseteq V(G)$ is a bipartite dominating set of graph G, if the subgraph $G[S]$ induced by S is bipartite and for every vertex not in S is dominated by any vertex in S. The bipartite domination number is denoted by $\gamma_{bp}(G)$ of graph G is the minimum cardinality of a

DOI: https://doi.org/10.71058/jodac.v9i8016

bipartite dominating set of G[2]. The upper bipartite domination number of G denoted as $\Gamma_{b\gamma}(G)$, is the maximum cardinality of a bipartite dominating set of G. In the area of chemistry, graph theory provides several useful tools such as topological indices. One of the most recent concepts is cheminformatics which integrates chemistry, mathematics, and information science Ahmed et.al [1].

Motivated from the concept domination index in graphs was introduced by Kavya and Sunitha [7]. In this paper we introduce the bipartite domination degree of a vertex. Topological indices are numerical values that quantity of graphs structural trails or attributes distance based topological indices and degree based topological indices are two of the most common types of topological indices Nadeem et.al [6]. Topological indices based on the degrees have a lot of significance.

The Wiener index $W(G)$ is the earliest and most well-known distance based index, established in 1947 by chemist Wiener [10]. Many additional distance based topological indices have been proposed and taken into consideration in chemical and mathematical chemistry literature such as Wiener index, Harary index [5] and eccentric connectivity index [9].

The Zagreb indices $M_1(G)$, $M_2(G)$ and $M_1^*(G)$ are the most frequently studied among them. These were introduced almost 40 years ago by Gutman et.al [4], Gutman and Trinajstic [3]:

$$M_1(G) = \sum_{x \in V(G)} d^2(x)$$

$$M_2(G) = \sum_{x,\, y \in E(G)} \left[d(x) \cdot d(y) \right]$$

$$M_1^*(G) = \sum_{x,\, y \in E(G)} \left[d(x) + d(y) \right]$$

2. Bipartite Domination Degree of a Graphs

In this section we introduce new degrees of a vertex say x using the minimal bipartite dominating set containing vertex x. The inequalities relating various graph parameters and the defined bipartite domination degree are obtained.

Definition 2.1: Let G be a graph and $x \in V$ then the bipartite domination degree of vertex x is defined as the minimum number of vertices in the minimal bipartite dominating set

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(MBDS) containing x, i.e $d_{b\gamma}(x)=min\{|D|$: D is a MDBS containing $x\}$. The degree of bipartite domination both minimum and maximum are defined as below:

$$\delta_{b\gamma}(G) = min\ \{d_{b\gamma}(x);\ x \in V(G)\} \text{ and } \Delta_{b\gamma}(G) = max\ \{d_{b\gamma}(x);\ x \in V(G)\}$$

Example 2.2: Consider the graph G in figure 1 below

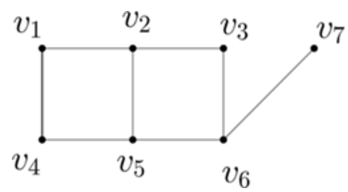

Figure 1:

Figure 1 illustration of bipartite domination degree of a vertex. Here $D_1=\{v_4, v_5, v_6\}$, $D_2=\{v_1, v_4, v_3, v_6\}$, $D_3=\{v_1, v_4, v_6, v_7\}$, $D_4=\{v_2, v_5, v_6, v_7\}$ are the bipartite dominating sets of G. Hence the MBDS containing v_4 with least number of vertices is $\{v_4, v_5, v_6\}$. Therefore $d_{b\gamma}(v_4) = 3$ similarly $d_{b\gamma}(v_5)= d_{b\gamma}(v_6) = 3$ and $d_{b\gamma}(v_1)=d_{b\gamma}(v_2)=d_{b\gamma}(v_3)=d_{b\gamma}(v_7) = 4$.

Defination 2.3: The first bdd zegreb, the second bdd and modified bdd zegreb indices are defined as:

$$D_{b\gamma}M_1(G) = \sum_{x\in V(G)} d_{b\gamma}^2(x)$$

$$D_{b\gamma}M_2(G) = \sum_{x,\ y\in E(G)} \left[d_{b\gamma}(x)\cdot d_{b\gamma}(y)\right]$$

$$D_{b\gamma}M_1^*(G) = \sum_{x,\ y\in E(G)} \left[d_{b\gamma}(x) + d_{b\gamma}(y)\right]$$

Proposition 2.4: Let G be a nontrivial connected graph then, $\gamma_{bp}(G) \leq d_{b\gamma}(x) \leq \Gamma_{bp}(G)$, $\forall x \in V(G)$

Proof: Follows by the definition of bipartite domination number, bipartite domination degree of a vertex and the upper bipartite domination number.

Observation 2.5: Let G be a nontrivial connected graph then $2\leq d_{b\gamma}(x)$ $\forall x \in V(G)$.

Proposition 2.6:Let H be a connected spanning subgraph of G then $d_{b\gamma}^G(x) \leq d_{b\gamma}^H(x)$ $\forall x \in V(G)$.

Corollary 2.6.1: Let T be a spanning tree of G then $d_{b\gamma}^G(x) \leq d_{b\gamma}^T(x)$ $\forall x \in V(G)$.

Proposition 2.7:For complete graph K_n, $n \geq 2$, $d_{b\gamma}(x) = 2$ $\forall x \in V(K_n)$.

Proof: Let $S \subseteq V(K_n)$ be a γ_{bp}-set of K_n. Let $V(K_n) = \{x_1, x_2, ..., x_n\}$ be the labeling of K_n. Now choose any two vertices x_i and x_j , $i \neq j$. Let $B = B_1 \cup B_2$, where $B_1=\{x_i\}$ and $B_2=\{x_j\}$. Clearly $K_n[B] = K_n[B_1 \cup B_2]$ is a bipartite graph and B is a bipartite dominating set. Thus, $|B| = |B_1| + |B_2| = 1 + 1 = 2$.

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Hence the MBDS containing x_i and x_j with least number of vertices is $\{x_i, x_j\}$. Therefore $d_{b\gamma}(x_i) = d_{b\gamma}(x_j) = 2$.

Theorem 2.8: Let G be a complete graph $K_n, n \geq 2$. Then,

$$D_{b\gamma}M_1(G) = 4n.$$

$$D_{b\gamma}M_2(G) = 2n(n-1).$$

$$D_{b\gamma}M_1^*(G) = 2n(n-1).$$

Proof:

$$\begin{aligned} D_{b\gamma}M_1(G) &= \sum_{x \in V(k_n)} d_{b\gamma}^2(x) \\ &= \sum_{x \in V(k_n)} 2^2 \\ &= 2^2 + 2^2 + \cdots + 2^2 \\ &= n \cdot 2^2 \\ &= 4n. \end{aligned}$$

$$\begin{aligned} D_{b\gamma}M_2(G) &= \sum_{x,y \in E(G)} \left[d_{b\gamma}(x) \cdot d_{b\gamma}(y) \right] \\ &= \frac{n(n-1)}{2} \left[(2) \cdot (2) \right] \\ &= 2n(n-1) \end{aligned}$$

$$\begin{aligned} D_{b\gamma}M_1^*(G) &= \sum_{x,y \in E(G)} \left[d_{b\gamma}(x) + d_{b\gamma}(y) \right] \\ &= \frac{n(n-1)}{2} \left[2 + 2 \right] \\ &= 2n(n-1) \end{aligned}$$

Proposition 2.9: For a complete bipartite graph, $K_{r,t}$, $d_{b\gamma}(x) = 2 \quad \forall x \in K_{r,t}$.

Proof: Let $x \in K_{r,t}$, then either $x \in V_1$ or $x \in V_2$, V_1 and V_2 are bipartite sets of $K_{r,t}$. Suppose $x \in V_1$. Then $\{x, y\}$ where y is any vertex of V_2 is a MBDS containing x with least number of vertices 2. Similarly, the case where $x \in V_2$. Hence $d_{b\gamma}(x) = 2$.

Theorem 2.10: Let $G = K_{r,t}$. Then,

$$D_{b\gamma}M_1(G) = 4n.$$

$$D_{b\gamma}M_2(G) = 4rt.$$

$$D_{b\gamma}M_1^*(G) = 4rt.$$

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Proof:

$$D_{b\gamma}M_1(G) = \sum_{x\in V(k_{r,t})} d_{b\gamma}^2(x)$$
$$= (r+t)2^2$$
$$= n\cdot 2^2$$
$$= 4n.$$

$$D_{b\gamma}M_2(G) = \sum_{x,y\in E(G)} \left[d_{b\gamma}(x)\cdot d_{b\gamma}(y)\right]$$
$$= r.t(2.2)$$
$$= 4rt$$

$$D_{b\gamma}M_1^*(G) = \sum_{x,y\in E(G)} \left[d_{b\gamma}(x) + d_{b\gamma}(y)\right]$$
$$= r.t\left[2+2\right]$$
$$= 4rt.$$

Proposition 2.11: For any wheel graph W_n, $n \geq 4$, $d_{b\gamma}(x) = 2\ \forall x \in W_n$.

Proof: Let B_1 and B_2 are two partite sets of W_n, where $B_1 = \{x : deg(x) = \Delta\}$ and $B_2 = V(W_n) - \{x\}$, choose one vertex from B_2 say y. Then clearly $\{x,y\}$ is a MBDS containing x contains 2 vertices. Hence $d_{b\gamma}(x) = 2\ \forall x \in W_n$.

Theorem 2.12: Let $G = W_n, n \geq 4$. Then,

$$D_{b\gamma}M_1(G) = 4n.$$

$$D_{b\gamma}M_2(G) = 8(n-1).$$

$$D_{b\gamma}M_1^*(G) = 8(n-1).$$

Proof:

$$D_{b\gamma}M_1(G) = \sum_{x\in V(G)} d_{b\gamma}^2(x)$$
$$= n\cdot 2^2$$
$$= 4n.$$

$$D_{b\gamma}M_2(G) = \sum_{x,y\in E(G)} \left[d_{b\gamma}(x)\cdot d_{b\gamma}(y)\right]$$
$$= (2n-2)(2.2)$$
$$= 4(2n-2)$$
$$= 8(n-1)$$

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$$
\begin{aligned}
D_{b\gamma}M_1^*(G) &= \sum_{x,y\in E(G)} \left[d_{b\gamma}(x)+d_{b\gamma}(y)\right] \\
&= (2n-2)\left[2+2\right] \\
&= 4(2n-2) \\
&= 8(n-1).
\end{aligned}
$$

Proposition 2.13: For any friendship graph F_n where $n \geq 2$, $d_{b\gamma}(x) = 2\ \forall x \in F_n$.

Proof: Let $G = F_n$ be a friendship graph on $2n+1$ vertices and $2n$ edges. Let B_1 and B_2 be two bipartite sets of F_n. Then $B_1 \cup B_2$, where $B_1 = \{x : deg(x) = \Delta\}$. Choose any one vertex y from $V(F_n) - \{x\}$. Clearly $\{x, y\}$ is a bipartite dominating set containing x, contains least number of vertices $\{x, y\}$. Hence $d_{b\gamma}(x) = 2$.

Theorem 2.14: Let $G = F_n, n \geq 1$. Then,

$$D_{b\gamma}M_1(G) = 4(2n+1).$$

$$D_{b\gamma}M_2(G) = 8n.$$

$$D_{b\gamma}M_1^*(G) = 8n.$$

Proof:

$$
\begin{aligned}
D_{b\gamma}M_1(G) &= \sum_{x\in V(G)} d_{b\gamma}^2(x) \\
&= (2n+1)\cdot 2^2 \\
&= 4(2n+1).
\end{aligned}
$$

$$
\begin{aligned}
D_{b\gamma}M_2(G) &= \sum_{x,y\in E(G)} \left[d_{b\gamma}(x)\cdot d_{b\gamma}(y)\right] \\
&= 2n\cdot(2.2) \\
&= 4\cdot(2n) \\
&= 8n
\end{aligned}
$$

$$
\begin{aligned}
D_{b\gamma}M_1^*(G) &= \sum_{x,y\in E(G)} \left[d_{b\gamma}(x)+d_{b\gamma}(y)\right] \\
&= (2n)\left[2+2\right] \\
&= 4(2n) \\
&= 8n.
\end{aligned}
$$

Proposition 2.15: For any firecracker graph $F_{m,n}$ where $n \geq 2$, $d_{b\gamma}(x) = 2m\ \forall x \in F_{m,n}$.

Proof : Let $G = F_{m,n}$ be a firecracker graph on mn vertices with $(mn-1)$ edges. Let S be a minimum bipartite dominating set of graph G. By the definition of the firecracker graph, the graph is obtained from the series of interconnected m copies of n stars by linking one leaf from each. Let $B_1 = \{v_1, v_2, v_3, \ldots, v_m\}$, be the set of all central vertices

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of $F_{m,n}$ and let $B_2=V(G)-B_1$, choose a set $X=\{u_1, u_2, u_3, \ldots, u_m\}$, where u_i adjacent to v_i, $i = 1, 2, 3, \ldots, m$. Hence $\{u_1, u_2, u_3, \ldots, u_m, v_1, v_2, v_3, \ldots, v_m\}$ is the MBDS containing x contains $2m$ vertices. Therefore, $d_{b\gamma}(x) = 2m$ $\forall x \in V(G)$.

Theorem 2.16: Let $G = F_{m,n}, n \geq 2$. Then,

$$D_{b\gamma}M_1(G) = 4m^3n.$$

$$D_{b\gamma}M_2(G) = 4m^2(mn-1).$$

$$D_{b\gamma}M_1^*(G) = 4m(mn-1).$$

Proof:

$$\begin{aligned} D_{b\gamma}M_1(G) &= \sum_{x\in V(G)} d_{b\gamma}^2(x) \\ &= (mn)\cdot(2m)^2 \\ &= 4m^3n. \end{aligned}$$

$$\begin{aligned} D_{b\gamma}M_2(G) &= \sum_{x,y\in E(G)} \left[d_{b\gamma}(x)\cdot d_{b\gamma}(y)\right] \\ &= (mn-1)\cdot(2m.2m) \\ &= 4m^2\cdot(mn-1). \end{aligned}$$

$$\begin{aligned} D_{b\gamma}M_1^*(G) &= \sum_{x,y\in E(G)} \left[d_{b\gamma}(x) + d_{b\gamma}(y)\right] \\ &= (mn-1)\left[2m+2m\right] \\ &= 4m(mn-1). \end{aligned}$$

Proposition 2.17: For any book graph B_k where $k \geq 1$,

$$d_{b\gamma}(x) = \begin{cases} 2 & \text{if } x \text{ is a centre vertex} \\ k+1 & \text{otherwise} \end{cases}$$

Theorem 2.18: Let $G = B_k, k \geq 1$ be the book graph. Then,

$$D_{b\gamma}M_1(G) = 8 + 2k(k+1)^2.$$

$$D_{b\gamma}M_2(G) = k^3 + 6k^2 + 5k + 4.$$

$$D_{b\gamma}M_1^*(G) = 4k^2 + 8k + 4.$$

Proof: Let book graph is cross product of S_{k+1} and P_2 with $2k+2$ vertices and $3k+1$ edges. In book graph (v, w_1) and (v, w_2) are adjacent to k distinct vertices each with the three types of edges, $B_{2,2}, B_{2,k+1}$ and $B_{k+1,k+1}$.

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$$D_{b\gamma}M_1(G)=\sum_{x\in V(G)} d_{b\gamma}^2(x)$$
$$=2\cdot 2^2+(2k)\cdot\left(k+1\right)^2$$
$$=8+2k(k+1)^2.$$

$$D_{b\gamma}M_2(G)=\sum_{x,y\in E(G)}\left[d_{b\gamma}(x)\cdot d_{b\gamma}(y)\right]$$
$$=1(2\cdot 2)+2k(2(k+1))+k((k+1)\cdot(k+1))$$
$$=k^3+6k^2+5k+4.$$

$$D_{b\gamma}M_1^*(G)=\sum_{x,y\in E(G)}\left[d_{b\gamma}(x)+d_{b\gamma}(y)\right]$$
$$=1(2+2)+2k(2+k+1)+k((k+1)+(k+1))$$
$$=4k^2+8k+4.$$

Theorem 2.19: [8]: Given a cycle C_n, $n\geq 3$. Then,

$$\gamma_{bp}(C_n)=\begin{cases}\frac{n}{2} & if\ n\equiv 0(\mathrm{mod}\,4)\\ \frac{n+1}{2} & if\ n\equiv 1,3(\mathrm{mod}\,4)\\ \frac{n+1}{2} & if\ n\equiv 2(\mathrm{mod}\,4)\end{cases}$$

Theorem 2.20: Let C_n be a cycle of order $n\geq 3$. Then,

(i) $$D_{b\gamma}M_1(G)=D_{b\gamma}M_2(G)=\begin{cases}\frac{n^3}{4} & if\ n\equiv 0(\mathrm{mod}\,4)\\ \frac{n(n+1)^2}{4} & if\ n\equiv 1,3(\mathrm{mod}\,4)\\ \frac{n(n+1)^2}{4} & if\ n\equiv 2(\mathrm{mod}\,4)\end{cases}$$

(ii) $$D_{b\gamma}M_1^*(G)=\begin{cases}n^2 & if\ n\equiv 0(\mathrm{mod}\,4)\\ n(n+1) & if\ n\equiv 1,3(\mathrm{mod}\,4)\\ n(n+2) & if\ n\equiv 2(\mathrm{mod}\,4)\end{cases}$$

Proof:B By Theorem 2.19, we have

$$d_{b\gamma}(x)=\begin{cases}\frac{n}{2} & if\ n\equiv 0(\mathrm{mod}\,4)\\ \frac{n+1}{2} & if\ n\equiv 1,3(\mathrm{mod}\,4)\\ \frac{n+1}{2} & if\ n\equiv 2(\mathrm{mod}\,4)\end{cases}\quad \forall x\in C_n$$

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If $n \equiv 0 (\bmod 4)$, then

$$\begin{aligned} D_{b\gamma} M_1(G) &= \sum_{x \in V(G)} d_{b\gamma}^2(x) \\ &= n \cdot \left(\frac{n}{2}\right)^2 \\ &= n \cdot \left(\frac{n}{2} \cdot \frac{n}{2}\right) \\ &= D_{b\gamma} M_2(G). \end{aligned}$$

If $n \equiv 1,3 (\bmod 4)$, then

$$\begin{aligned} D_{b\gamma} M_1(G) &= \sum_{x \in V(G)} d_{b\gamma}^2(x) \\ &= n \cdot \left(\frac{n+1}{2}\right)^2 \\ &= n \cdot \left(\frac{n+1}{2} \cdot \frac{n+1}{2}\right) \\ &= D_{b\gamma} M_2(G) \end{aligned}$$

If $n \equiv 2 (\bmod 4)$, then

$$\begin{aligned} D_{b\gamma} M_1(G) &= \sum_{x \in V(G)} d_{b\gamma}^2(x) \\ &= n \cdot \left(\frac{n+2}{2}\right)^2 \\ &= n \cdot \left(\frac{n+2}{2} \cdot \frac{n+2}{2}\right) \\ &= D_{b\gamma} M_2(G). \end{aligned}$$

$$D_{b\gamma} M_1^*(G) = \sum_{x, y \in E(G)} \left[d_{b\gamma}(x) + d_{b\gamma}(y) \right]$$

If $n \equiv 0 (\bmod 4)$, then

$$\begin{aligned} D_{b\gamma} M_1^*(G) &= n \cdot \left[\frac{n}{2} + \frac{n}{2}\right] \\ &= n^2. \end{aligned}$$

If $n \equiv 1,3 (\bmod 4)$, then

$$\begin{aligned} D_{b\gamma} M_1^*(G) &= n \cdot \left[\frac{n+1}{2} + \frac{n+1}{2}\right] \\ &= n(n+1). \end{aligned}$$

If $n \equiv 2 (\bmod 4)$, then

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$$D_{b\gamma}M_1^*(G) = n \cdot \left[\frac{n+2}{2} + \frac{n+2}{2}\right]$$
$$= n(n+2).$$

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