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A SYMMETRIC INFORMATION MEASURE AND INNOVATIVE BOUNDS IN THE THEORY OF COMMUNICATION SYSTEM

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Abstract: *A A non-parametric theory-based innovative symmetric information divergence measure for communication systems is proposed. This information measure belongs to the family of f-divergence. Furthermore, for the first time, we derive some bounds i.e. equalities and inequalities for this information divergence measure with well-known divergence measures, namely: Symmetric Chi-Square divergence Measure, Kumar & Johnson divergence and Bhattacharyya information measure, based on two distinct discrete probability distributions*

2020 Mathematics Subject Classification*: 94A05; 94A17; 26D15.*

Keywords and Phrases*: Communication system, non-parametric divergence measure, Symmetric Chi-Square divergence Measure, Csiszár's f-divergence, Arithmetic mean divergence, Triangular discrimination.*

1. Introduction

Let $\Gamma_n = \{P = (p_1, p_2, p_3, \ldots p_n);\ p_i > 0, \sum_{i=1}^{n} p_i = 1\}$, n ≥2, be the set of all complete finite discrete probability distributions. Csiszár [2, 3] introduced a generalized measure of information using the *f*-divergence measure

$$C_f(P,Q) = \sum_{i=1}^{n} q_i\, f\left(\frac{p_i}{q_i}\right) \tag{1.1}$$

Where $f:(0,\infty) \to R$ (set of real numbers) is a convex function. Most common choices of function *f* satisfy *f* (1) = 0, so that. $C_f(P,Q) = 0$. The convexity of the function *f* ensures that the divergence measure $C_f(P,Q)$ is nonnegative. An important characteristic of this divergence measure is that many known divergences [4, 5, 6, 12], can be obtained from this measure by appropriately defining the convex function *f*.

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Furthermore, Information divergence measures are important in communication systems as they quantify the variance between probability distributions, empowering the assessment of communication channel performance, error rates, and the effectiveness of data compression techniques. They help in understanding how much information is lost or gained during transmission, aiding in designing more reliable and efficient communication systems.

The edifice of an information divergence measure for two distinct probability distributions is not an easy task.

Many authors introduced information measures, some of them are as follows: -

- Chi-square divergence (Person, [8])

$$\chi^2(P,Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{q_i} \tag{1.2}$$

- Symmetric Chi-square divergence (Dragomir et al., [20])

$$\psi(P,Q) = \chi^2(P,Q) + \chi^2(Q,P) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2 (p_i + q_i)}{p_i q_i} \tag{1.3}$$

- Triangular discrimination (Topose, [14])

$$\Delta(P,Q) = 2[1 - H(P,Q)] = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{p_i + q_i} \tag{1.4}$$

- Harmonic mean divergence (Taneja, [18])

$$H(P,Q) = \sum_{i=1}^{n} \frac{2\, p_i\, q_i}{p_i + q_i} \tag{1.5}$$

- Kumar and Johnson divergence (Kumar and Johnson, [4])

$$\psi_M(P,Q) = \sum_{i=1}^{n} \frac{(p_i^2 - q_i^2)^2}{2(p_i\, q_i)^{3/2}} \tag{1.6}$$

- Bhattacharyya information measure (Bhattacharyya, [15])

$$B(P,Q) = \sum_{i=1}^{n} \sqrt{p_i\, q_i} \tag{1.7}$$

In this research work, we are introducing a new theory-based non-parametric information divergence measure which fits into the category of Csiszár's $f -$Divergences [2, 3].

A different theory-based non-parametric symmetric information divergence measure is achieved in section 2. In section 3, we obtained some unique propositions (equalities and inequalities) for the new information divergence measure in terms of some recognized and valued divergence measures. Section 4 concludes the paper.

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Note: For shortness, we will denote p_i, q_i and $\sum_{i=1}^{n}$ by p, q and $\sum$ respectively.

2. New Information Divergence Measure

Now, we consider the function $f:(0,\infty)\to\mathbb{R}$, given by

$$f(u)=\frac{8\,(u-1)^2\left(u^2+1\right)}{(u+1)^3} \tag{2.1}$$

Next,

$$f^{/}\,(t)=\frac{8\,(u-1)\left(u^3+5u^2-3u+5\right)}{(u+1)^4} \tag{2.2}$$

And

$$f''(u)=\frac{32\left(7u^2-10u+7\right)}{(u+1)^5} \tag{2.3}$$

The function f (u) is convex since $f''(u)>0$ for all $u > 0$ and normalized also since $f(1) = 0$.

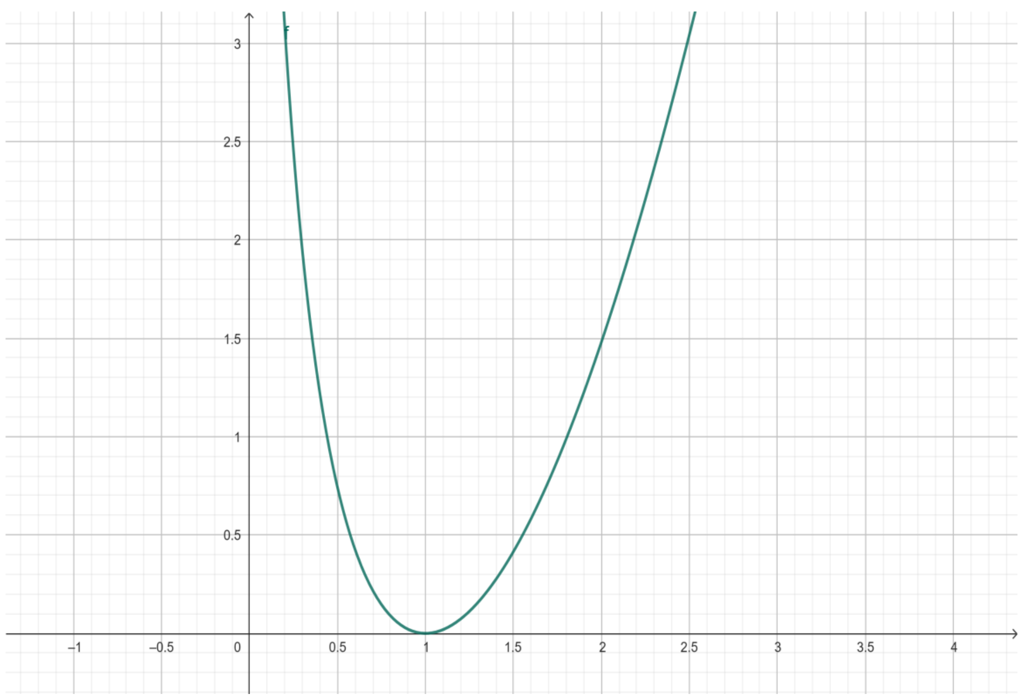

Figure 1: Graph of the function f (u)

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Figure 1 shows the behavior of the function $f(u)$ and which is continuously convex.

Ali- Silvey [1] and Csiszár's [2, 3] introduced the generalized measure of information using the *f*-divergence measure given by

$$C_f(P,Q) = \sum_{i=1}^{n} q_i \, f\left(\frac{p_i}{q_i}\right) = \sum q \, f\left(\frac{p}{q}\right) \tag{2.4}$$

Applying Csiszár's *f*-divergence properties on equation (2.1), we get

$$\mathrm{N}(P,Q) = 8\sum_{i=1}^{n} \frac{(p_i-q_i)^2(p_i^2+q_i^2)}{(p_i+q_i)^3} = 8\sum \frac{(p-q)^2(p^2+q^2)}{(p+q)^3} \tag{2.5}$$

New information measure $N(P,Q)$has the following properties:-

a) $N(P,Q) = 0$ *for* $P = Q$ (2.6)

b) $N(P,Q)$ is a symmetric information measure, since

$$\mathrm{N}(P,Q) = \mathrm{N}(Q,P) \tag{2.7}$$

c) The measure is nonnegative in the pair of discrete probability distributions. $(P,Q) \in \Gamma_n$.

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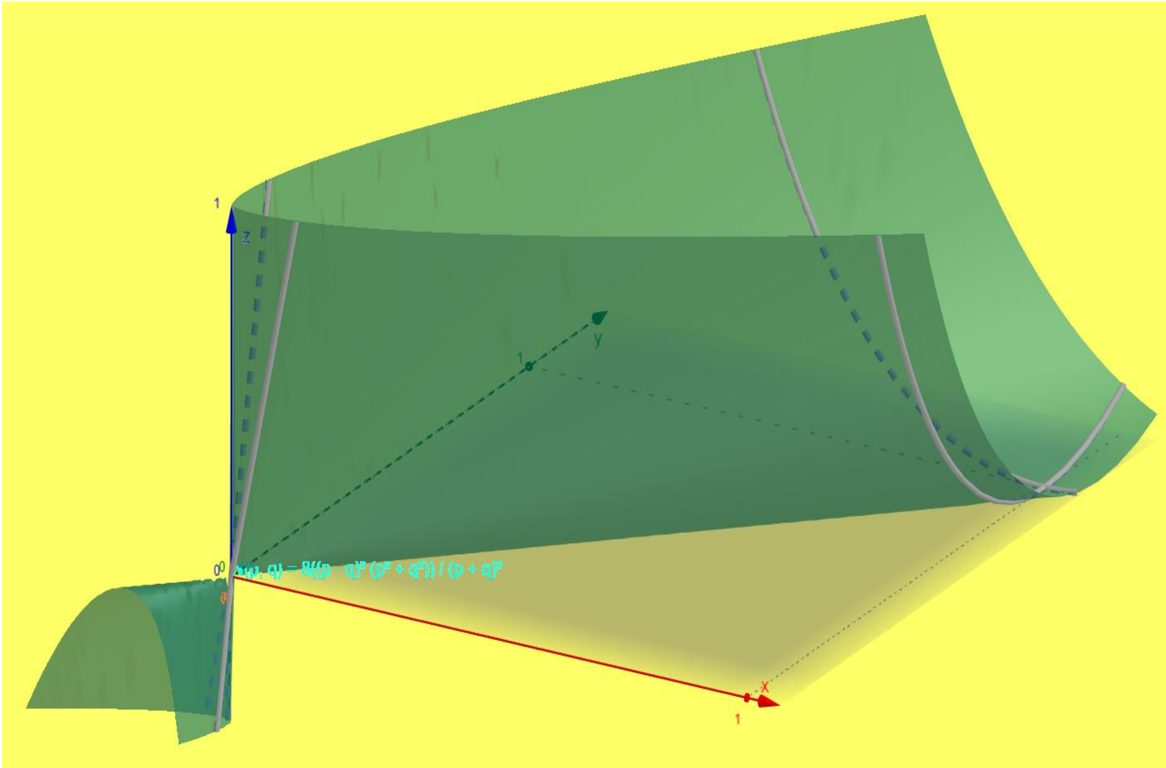

Figure2: 3D- Presentation of new information Measure N(P, Q)

3. Some Equalities and Inequalities of New Information Measure With other Recognized Measures

In this section, we now derive some information equalities and inequalities for *N(P, Q)* with other recognized information measures in the following propositions.

Proposition 3.1: Let$(P, Q) \epsilon \Gamma_n \times \Gamma_n$, then we have the following new equalities

$$N(P,Q) = 4\,\Delta(P,Q)[2 - H(P,Q)] \tag{3.1}$$

and

$$N(P,Q) = 8\,[1 - H(P,Q)][2 - H(P,Q)] \tag{3.2}$$

Where $H(P,Q)\ and\ \Delta(P,Q)$ are given by (1.5) and (1.4) respectively.

Proof: From (2.5), we have a new information measure,

$$\mathrm{N}(P,Q) = 8\sum_{i=1}^{n}\frac{(p_i-q_i)^2(p_i^2+q_i^2)}{(p_i+q_i)^3}$$

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$$= 8\sum \frac{(p-q)^2(p^2+q^2)}{(p+q)^3}$$

$$= 8\sum \frac{(p-q)^2}{(p+q)} \sum \frac{(p^2+q^2)}{(p+q)^2}$$

$$= 8\sum \frac{(p-q)^2}{(p+q)} \sum \frac{\{(p^2+q^2)+2pq\}-2pq}{(p+q)^2}$$

$$= 8\sum \frac{(p-q)^2}{(p+q)} \left[\sum \frac{\{(p+q)^2-2pq\}}{(p+q)^2}\right]$$

$$= 8\sum \frac{(p-q)^2}{(p+q)} \left[1 - \sum \frac{2pq}{p+q} \sum \frac{1}{p+q}\right]$$

Using (1.4) and (1.5), we get

$$N(P,Q) = 8\,\Delta(P,Q)\left[1 - H(P,Q)\frac{1}{2}\right]$$

$$N(P,Q) = 4\,\Delta(P,Q)[2 - H(P,Q)]$$

Again, using (1.4), we get the result.

$$N(P,Q) = 8\,[1 - H(P,Q)][2 - H(P,Q)]$$

hence the result.

Proposition 3.2: Let$(P,Q) \epsilon \Gamma_n \times \Gamma_n$, then we have the following new inequality

$$N(P,Q) \le 4\,\Delta(P,Q) \tag{3.3}$$

and

$$N(P,Q) \le 8[1 - H(P,Q)] \tag{3.4}$$

Where $H(P,Q)$ *and* $\Delta(P,Q)$ are given by (1.5) and (1.4) respectively.

Proof: From (2.5), we have a new information measure,

$$\mathrm{N}(P,Q) = 8\sum_{i=1}^{n} \frac{(p_i-q_i)^2(p_i^2+q_i^2)}{(p_i+q_i)^3}$$

$$= 8\sum \frac{(p-q)^2(p^2+q^2)}{(p+q)^3}$$

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$$= 8\sum\frac{(p-q)^2}{(p+q)}\sum\frac{(p^2+q^2)}{(p+q)^2}$$

$$N(P,Q) \le 8\sum\frac{(p-q)^2}{(p+q)}\sum\frac{(p+q)}{(p+q)^2}\ ;$$

$$Since\ p^2+q^2\ \le p+q$$

$$=\ 8\sum\frac{(p-q)^2}{(p+q)}\sum\frac{1}{p+q}$$

$$=\ 8\sum\frac{(p-q)^2}{(p+q)}\ \frac{1}{2}$$

Using (1.4), we get

$$N(P,Q) \le 4\,\Delta(P,Q)$$

And $$N(P,Q) \le 8[1-H(P,Q)]$$

hence the result.

Proposition 3.3: Let$(P,Q)\,\epsilon\,\Gamma_n\ \times\Gamma_n$, then we have the following new inequality

$$N(P,Q)\ \le \Psi_M(P,Q)\ \ [H(P,Q)]^4\ \left[\frac{1}{B(P,Q)}\right]^5 \tag{3.5}$$

Where $H(P,Q), \psi_M(P,Q)$ and *B(P, Q)* are given by (1.5), (1.6), and (1.7) respectively.

Proof: From (2.5), we have a new information measure,

$$\mathrm{N}(P,Q)\ =\ \ 8\sum_{i=1}^{n}\frac{(p_i-q_i)^2(p_i^2+q_i^2)}{(p_i+q_i)^3}$$

$$= 8\sum\frac{(p-q)^2(p^2+q^2)}{(p+q)^3}$$

$$= 8\sum\frac{(p-q)^2\,(p+q)^2}{2(p\,q)^{3/2}}\ \frac{2(p\,q)^{3/2}}{(p+q)^2}\ \frac{(p^2+q^2)}{(p+q)^3}$$

$$\le 4\sum\frac{(p^2-q^2)^2}{2(p\,q)^{3/2}}\ \left[\sum\frac{2\,p\,q}{p+q}\right]^2\sum\frac{(p+q)}{(p+q)^3}\ \frac{1}{\sqrt{p\,q}}$$

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Using (1.5) and (1.6), since $p^2+q^2 \leq p+q$

$$= 4\,\Psi_M(P,Q)\ \ [H(P,Q)]^2 \sum \frac{1}{(p+q)^2}\,\frac{1}{\sqrt{p\,q}}$$

$$= \Psi_M(P,Q)\ \ [H(P,Q)]^2 \left[\sum \frac{2\,p\,q}{p+q}\right]^2 \left[\sum \frac{1}{\sqrt{p\,q}}\right]^5$$

Again, using (1.5) and (1.7), we get

$$N(P,Q)\ \leq \Psi_M(P,Q)\ \ [H(P,Q)]^4 \left[\frac{1}{B(P,Q)}\right]^5$$

Hence the result.

Proposition 3.4: Let$(P,Q)\ \epsilon\ \Gamma_n\ \times \Gamma_n$, then we have the following new inequality.

$$N(P,Q) \leq\ [\chi^2(P,Q)+\chi^2(Q,P)][H(P,Q)]^3 \left[\frac{1}{B(P,Q)}\right]^4 \tag{3.6}$$

Where $\psi(P,Q)=\chi^2(P,Q)+\chi^2(Q,P),\ H(P,Q)$ and $B(P,Q)$ are given by (1.3), (1.5), and (1.7), respectively.

Proof: From (2.5), we have a new information measure,

$$\mathrm{N}(P,Q)\ =\ \ 8\sum_{i=1}^{n}\frac{(p_i-q_i)^2(p_i^2+q_i^2)}{(p_i+q_i)^3}$$

$$= 8\sum \frac{(p-q)^2(p^2+q^2)}{(p+q)^3}$$

$$=\ 4\sum \frac{(p-q)^2(p+q)}{p\,q} \sum \frac{2\,p\,q}{p+q} \sum \frac{p^2+q^2}{(p+q)^3}$$

We know that $p^2+q^2 \leq p+q$, so we get

$$N(P,Q) \leq\ 4\sum \frac{(p-q)^2(p+q)}{p\,q} \sum \frac{2\,p\,q}{p+q} \sum \frac{p+q}{(p+q)^3}$$

$$=\ \sum \frac{(p-q)^2(p+q)}{p\,q} \sum \frac{2\,p\,q}{p+q} \left[\sum \frac{2\,p\,q}{p+q}\right]^2 \left[\sum \frac{1}{\sqrt{p\,q}}\right]^4$$

Now using (1.3), (1.5), and (1.7), we get

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$$N(P,Q) \leq [\chi^2(P,Q) + \chi^2(Q,P)][H(P,Q)]^3 \left[\frac{1}{B(P,Q)}\right]^4$$

hence the inequality.

4. Conclusion

During past years, Teneja [11, 12, 18], Dragomir [20], Kumar and others [4, 5] gave the idea of information measures, their properties, bounds, and relations with other measures. Kumar and others did a lot of work, especially in the field of information theory. We have introduced a new innovative theoretical base nonparametric symmetric information measure, in the Csiszár's f-divergence category [2, 3], by considering a convex function f, defined on (0, ∞). In this paper, we also define the very useful propositions of the new innovative information measure with the Symmetric Chi-Square divergence Measure, Kumar & Johnson divergence and Bhattacharyya information measure, based on two distinct discrete probability distributions.

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