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BAYESIAN INFERENCE ANALYSIS OF RANDOM OCCURRENCE OF IDLE SERVERS OF MARKOVIAN QUEUEING MODELS

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Abstract: The paper focuses on the study of random occurrence of idle servers which is an imperative phenomenon connected with queueing models. Such a kind of study leads to proper utilization of servers' working time in the system. Various analytics related to Bayesian approach to single and multi-server Markovian queueing models have been proposed to figure out and discuss in the paper. Bayesian inferential analysis has been so far attracted insufficiently by researchers engaged in this field despite being versatile, potential and appropriate method to effectively deal with uncertainty environment embedded with queueing models wherever estimation of parameters involved in probability distributions is intended. R software has been used to compute and perform the analytics of highest posterior density, credible interval and detection of random occurrence of idle server related to Bayesian inference of random occurrence of idle servers of the Markovian queueing model under study.

Keywords: Markovian queue, Bayesian inference, random occurrence of idle servers, highest posterior density, credible interval

Mathematics Subject Classification 2020: 00A05, 90B22, 60K25, 60K30, 68M20

1. Introduction

Queuing theory involves the mathematical study of queues or waiting lines. The representation of waiting lines (or queues) is a common phenomenon that occurs when the current demand for a service exceeds the current capacity to deliver the service and demands of customers are expected to be fulfilled simultaneously. Another technical reason of formation of queue is existence of at least two contradictory costs of waiting and service costs of the system, vide as Bunday [11], Taha [40], Gross and Harris [28], Singh and Kumar [38].

Flow of customers with finite and infinite population from one queue (waiting line) towards service facility due to lack of capacity to serve them all at one time. In the absence of a right balance between service facility and customers, waiting times are required for service facilities or for the arrival of customers; vide Kendall [32], Hiller and Liebermann [29]. An approach of computational to cost and profit analysis of clocked queuing networks used by Mishra, Yadav & Shukla [33, 34].

Here, we consider Markovian queueing model of which both arrival and service follow Poisson probability law and it has single server with first come and first served discipline as well as infinite capacity. There are several operating characteristics or performance measures of the queuing model namely traffic congestion, expected number of customers in queue and system; expected waiting time in queue and system; and service utilization factor or busy period. Arrival and service events are probabilistic whereas mean and service arrival rates as numerical expressions are possibilistic. Here, vide as Bhattacharya and Singh [9], Choudhury and Basak [6, 7] Chowdhury and Mukherjee [19, 20], Dieleman [25]. The queueing models have following popular features, vide Armero and Bayarri [1 2,3], Basak and Choudhury [16], Cruz et al. [22, 39].

The application of queuing models is well known for making effective management decisions which include as jobs waiting for processing; reduction of congestion in traffic; parts waiting in assembly lines; inventory control problems etc., vide Chowdhury and Mukherjee [20, 19], Kendall [32], Bernardo [8], Charles and Michael [14], Deepthi and Jose [15]. It also ensures appropriate and good understanding about the phenomenon of waiting lines so that adequate service can be provided with affordable waiting; it can determine and control arrivals and optimal number of service providers, vide as Marin and Robert [22, 39], Mishra, Shukla and Yadav [33, 34], Braham et

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al. [10]. It becomes difficult for mathematical analysis, vide as Gross and Harris [28], Hillier and Liebermann [29], Sutton and Jorden [14].

Bayesian inference is a branch of inference. However, its role and influence on the development of statistical inference is much deeper. Its philosophical basis dates back to the very early and rather subjective interpretation of the notion of probability during the Hellenistic period (323–146 BCE). Nowadays, Bayesian statistics are regarded as the study of uncertain phenomena through the notion of probability. It aims to develop a coherent methodology for inductive mathematical reasoning. Bayesian inference is applied to a wide variety of statistical problems, including inference theory, hypothesis testing, model selection, and hierarchical models, vide as Bernardo and Smith [8], Donovan and Mickey [26], Choudhury at al. [17, 18], Singh and Kumar [38]. Bayesian analysis for the approach to statistical inference characterizing key idea: all unknown quantities including the parameters of random variables with probability distributions used to describe the state of knowledge about the values of these unknowns. Also statistical conclusions about unknown quantities based on observed data are obtained using the Bayes theorem, vide as Armero and Bayarri [1, 2, 3], Gelman et al. [27], David et al. [23], Singh et al. [39].

Qualitatively, the Bayesian approach to estimation of population begins with a probability distribution describing the state of knowledge about unknown quantities (usually parameters) before collecting data, and then to update this distribution using observed data. This article reviews the basic elements of a Bayesian analysis (model specification, computation of posterior distributions, model checking, and sensitivity analysis). Additional sections address the choice of prior distributions, and the application of Bayesian methods, vide Bhattacharyya and Singh [9], Cruz et al. [22], Dey [24], Joby [31], Mukherjee and Chowdhury [19, 20], Yasushi and Naoki [42], Bura and Sharma [12], Basak and Choudhury [6, 7, 13, 16].

Typically, Bayesian methods are data analysis tools derived from the principles of Bayesian inference. In addition to their formal interpretation as a means of induction, Bayesian methods provide- estimation of parameters with good statistical properties; remuneration details of observed data; forecasting for missing data and forecasting future data; a computational framework for model estimation; selection and validation. Thus the uses of Bayesian methods go beyond the formal task of induction for which the methods are derived. Throughout this concept we will explore the broad uses of Bayesian methods for a variety of inferential and statistical tasks, vide Bernardo [8], Joby [31], Yasushi and Naoki [42]. A study of a two-component system with common-cause shock failures using Bayesian approach has been introduced by Insua et al. [27]. Spatial statistics and Bayesian computation approached given by Wiper [41]. A fundamental approach to Bayesian Inference in econometrics has been developed by Seghal and Agrawal [37], Congdon [21], Gelman et al. [27]. A review to developed short introduction to applied Bayesian statistics has been discussed by Donovan and Mickey [26].

In the present paper, a background and justification of the problem under consideration are given as follow. As we know that statistical analysis plays an important role in the analysis of queueing models connected with queueing theory. There is an interesting aspect of queueing theory that it may be an arrival or departure pattern of the stochastic queueing system; probability distributions characterize the arrival as well as departure of the customers engaged with service channel. Here, probability distributions representing arrival and departure involve population parameters. These parameters are generally computed by using optimization and other computational processes; and these values are substituted in the further computation of performance measures of the queueing system. In this context, a point becomes very imperative to enlighten that this population parameter can be appropriately obtained through the process of estimation only. Therefore, theory of estimation must enter into this picture to realistically obtain the value of parameters involved in probability distributions which are used in queueing system. Moreover, in the theory of estimation, there are two schools- one is frequentist and another is Bayesian. In frequentist approach, there is no consideration of prior distribution and thus is a probabilistic statement about the interval, known as confidence interval in which true value of parameter may fall or not. But in case of Bayesian approach, this is a probabilistic statement about the location of the true value of population parameter in the given interval, known as credible interval in which true value lies.

In Bayesian inference, an interesting and important point is consideration of prior distribution which represents past information about the population as belief and also it considers new evidence from the present observations of the event (likelihood); making thus a joint and effective information apparatus based on past and new information popularly known as posterior distribution intending to provide a more realistic estimated value of parameter of given distribution. Keeping this in mind as very less researches reported in this area, Bayesian

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inference has been applied to the queueing model under consideration to extract the population parameter by using R computing through large number of iterations of the process. Moreover, no previous problem in the present form has been attempted by researchers engaged in this field. This certainly motivates us to proceed in the direction to bridge this critical gap of research.
The proposed problem has two parts. One part is analyzed for single-server Markovian queueing model and another part is analyzed for multi-server Markovian queueing model. In the single-server model, server is randomly observed for a certain number of times and occurrence of idleness is reported to generate a sample data. Likewise, multi servers are randomly observed for their individual idleness and accordingly another sample data is generated for further analysis of Bayesian inference using R to provide the highest posterior density, probability of ascertainment of random occurrence of idle server and its credible interval for the model as important results connected with the problem. The paper aims to include important sections such as introduction, queueing models, mechanism of Bayesian inference, model analysis, and algorithm for R, implementation of algorithm and conclusion.

2. Material and Methods

We have the following important parts which are explained one by one.

3. Notations and Assumptions

3.1 Notations

Notations used frequently are the following.
λ : Average Arrival rate of customer.
μ : Average Service rate.
C_n : Normalizing constant (Evidence)
$L_n(\theta)$: Likelihood function
$\pi(\theta)$: Prior Distribution.
$p(x|\theta)$: Conditional probability density

3.2 Assumptions

The following assumptions are

(i) Random occurrence of idle server follows Binomial Distribution.
(ii) Number of observations to identify and detect random occurrences is independent and finite.

4. Mechanism of Bayesian Inference

Let n sample observations $X_1, X_2, X_3 \ldots \ldots \ldots \ldots.., X_n$ be drawn from a probability density $p(x|\theta)$. In this paper, we write $p(x|\theta)$ if we view θ as a random variable and $p(x|\theta)$ represents the conditional probability density of X conditioned on θ. In contrast, we write $p_\theta(X)$ if we view θ as a deterministic value. Bayesian inference is usually carried out in the following way.

i. We choose a probability density $\pi(\theta)$ – called the prior distribution – that expresses our beliefs about a parameter θ before we see any data.

ii. We choose a statistical model $p(x|\theta)$ that reflects our beliefs about x given θ.

iii. After observing data $D_n = \{X_1, X_2, X_3 \ldots \ldots \ldots \ldots.., X_n\}$, we update our beliefs and

Calculate the posterior distributionp$(\theta|D_n)$.

iv. By Bayes' theorem, the posterior distribution can be written as

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$$p(\theta|X_1, X_2, X_3 \ldots\ldots\ldots\ldots.., X_n) = \frac{(p(X_1, X_2, X_3 \ldots\ldots\ldots\ldots.., X_n)|\theta)\pi(\theta)}{p(X_1, X_2, X_3 \ldots\ldots\ldots\ldots.., X_n)}$$

$$= \frac{L_n(\theta)\pi(\theta)}{C_n} \propto L_n(\theta)\pi(\theta)$$

where $L_n(\theta) = \prod_{i=1}^{n} p(X_i|\theta)$ is the likelihood function and

$C_n = p(X_1, X_2, X_3 \ldots\ldots\ldots\ldots.., X_n) = \int (X_1, X_2, X_3 \ldots\ldots\ldots\ldots.., X_n|\theta)\ \pi(\theta)d\theta$

$= \int L_n(\theta)\ \pi(\theta)d\theta$ is the normalizing constant, which is also called the evidence.

We can get a Bayesian point estimate by summarizing the centre of the posterior. Typically, we use the mean or made or made of the posterior distribution. The posterior mean is $\bar{\theta}_n = \int \theta p(\theta|D_n)\, d\theta = \frac{\int \theta L_n(\theta)\pi(\theta)d\theta}{\int L_n(\theta)\pi(\theta)d\theta}$

We can also obtain a Bayesian interval estimate. For example, for $\alpha\epsilon(0,1)$, could find a and b such that $\int_{-\infty}^{a} p(\theta|D_n)d\theta = \int_{b}^{\infty} p(\theta|D_n)d\theta = \frac{\alpha}{2}$

Let $C = (a, b)$. Then $\mathbb{P}(\theta\epsilon C|D_n)d\theta = \int_{a}^{b} p(\theta|D_n)d\theta = 1 - \alpha$,

So C is a $1 - \alpha$ Bayesian posterior interval or credible interval. If θ has more than one dimension, the extension is straightforward and we obtain a credible region.

5. Model Analysis

Model analysis is mainly based on the concept of likelihood function which is described in the following manner to use it in the model analysis.
Likelihood function plays very important role while estimating unknown parameters of the given probability distribution. Some other classical and simple methods are to name as minimum chi-square (for count) and method of moments. Estimators of such kind lack efficiency and their sampling probability distributions are not easy to track.
Let random variables $X_1, X_2, \ldots\ldots\ldots., X_n$ follow $f(X_1, X_2, \ldots\ldots, X_n|\theta)$ as joint probability distribution. Let us assume the random variables take values as $X_1 = x_1, X_2 = x_2 \ldots\quad X_n = x_n$, then function of θ is defined by
$L(\theta) = L(\theta|x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n|\theta)$ is known as likelihood function.
As we know that likelihood function is a common component of Bayesian and frequentist analyses and it is not considered as a probability density-function. If we have $L(\theta_1|x) > L(\theta_2|x)$ for two parameters θ_1 and θ_2 then for $\theta = \theta_1$, θ1 is found a more suitable value for θ as compared to θ_2.
In the case of Binomial distribution with parameters n and p
$f(x, p) = C_x^n\, p^x(1-p)^{n-x}, \quad x = 0,1,2,3 \ldots\ldots n.$
Now, Beta Distribution is given as $f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$, where
$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\, dx$
Then, prior is considered as $\pi(p) = \frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha,\ \beta)}$ where $B(\alpha, \beta) = \int_0^1 p^{\alpha-1}(1-p)^{\beta-1}\, dp$
Here, it is interesting to note that parameter p of binomial distribution has become random variable in prior considered for the purpose.
Likelihood function is further defined as

$$L_n(p) = \prod_{i=1}^{n} P(x_i|p)$$

$L_n = \prod_{i=1}^{n} C_{x_i}^n\, p^{x_i}q^{n-x_i} = C_{x_1}^n\, p^{x_1}q^{n-x_1} * C_{x_2}^n\, p^{x_2}q^{n-x_2} * \ldots\ldots\ldots..* C_{x_n}^n\, p^{x_n}q^{n-x_n}$
$C_n = \int L_n(p) * \pi(p)\, dp$ and posterior is defined as product of likelihood and prior
That is, Posterior = Likelihood * Prior. This implies that

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Posterior = $\prod_{i=1}^{n} C_{x_i}^{n}\, p^{x_i} q^{n-x_i} * \frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha,\beta)}$ because C_n as an integration value which comes out as constant. This expression of posterior plays important role in the analysis of Bayesian inference

6. Algorithm for R

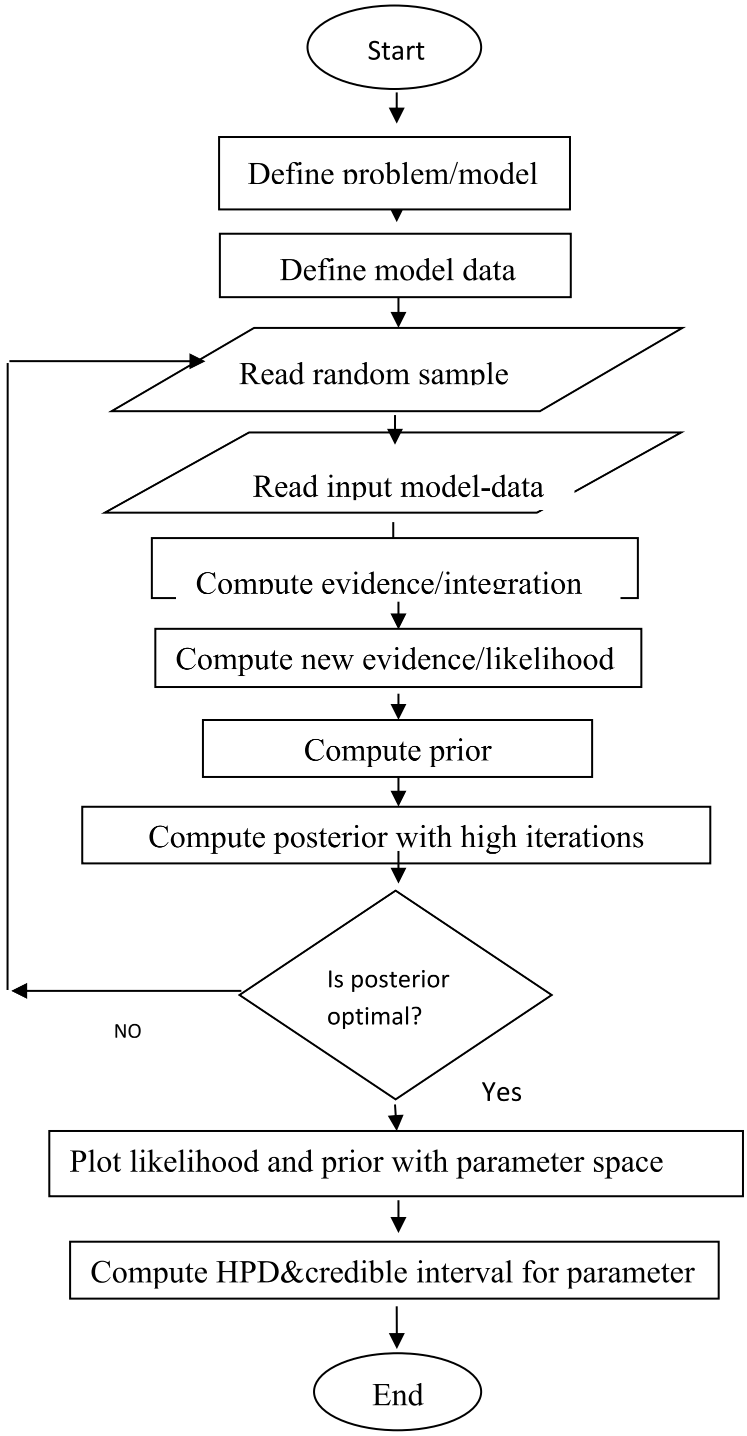

Fig 1 Flow Chart of Computing Algorithm for R

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7. Extraction of Parameter

Here, extraction of parameter is made through R software. R has double edged advantages. First, it is language and second, it is an environment too. It uses number of iterations as optimal iterations to extract the parameters of posterior distribution. It has enormous features, vide as [35, 36], Albert Jim [4], Albert, J. [5].

8. Bayesian Inference Analysis

Assuming that let total random observations of single server in a single-server model be n and reported number of idleness found be x. The sample data is (n, x). Here, bionomial probability distribution is assumed to apply in Bayesian inference analysis. For numerical and computational purpose, particular sample data is considered as (11, 3). Further, assuming that let there be N servers in a multi-server model and each server is randomly observed by n times and each one's idleness is denoted by x_i, i = 1, 2, 3,....,N. The sample data is {(n, x_1), (n, x_2), (n, x_3), (n, x_4), ..., (n, x_N)}. For computational purpose, particular sample data for 12 servers is considered as {(11, 4), (11, 6), (11, 4), (11, 7), (11, 5), (11, 3), (11, 8), (11, 9), (11, 4), (11, 2), (11, 4), (11, 3)}. Now, graphic observations of Bayesian inference are presented as below.

9. Implementation of Algorithm

Algorithm now has been executed and following results are obtained.

9.1 Conjugate Prior

A prior becomes conjugate when both of distributions –prior, posterior can belong to the same family of distributions. For example, prior and posterior distributions belong to the same families such as gamma, beta, normal etc. Now, a question may arise here that why do we choose prior as beta distribution? In this regard, a simple answer lies in the fact that the beta-distribution is **conjugate prior** for parameter *p* of binomial distribution and posterior as discussed in the section of model analysis becomes of beta type distribution. In other words, prior and posterior distributions both belong to beta type family. This makes Bayesian estimation easy and straightforward.

9.2 Analysis for Queueing Model

The following analysis is presented for queueing model.

9.3 Informative Prior

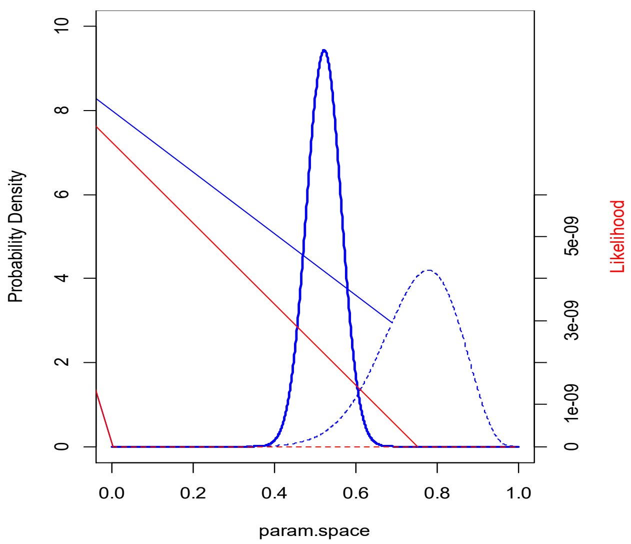

Fig 2: Graph of Informative Prior

In the above figure 2, likelihood value comes to near zero in red and posterior curve is depicted in dotted blue is bell shaped less peaked as compared to prior curve which is highly peaked in blue. Both are shifted and they don't coincide.

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9.4 Very Informative Prior

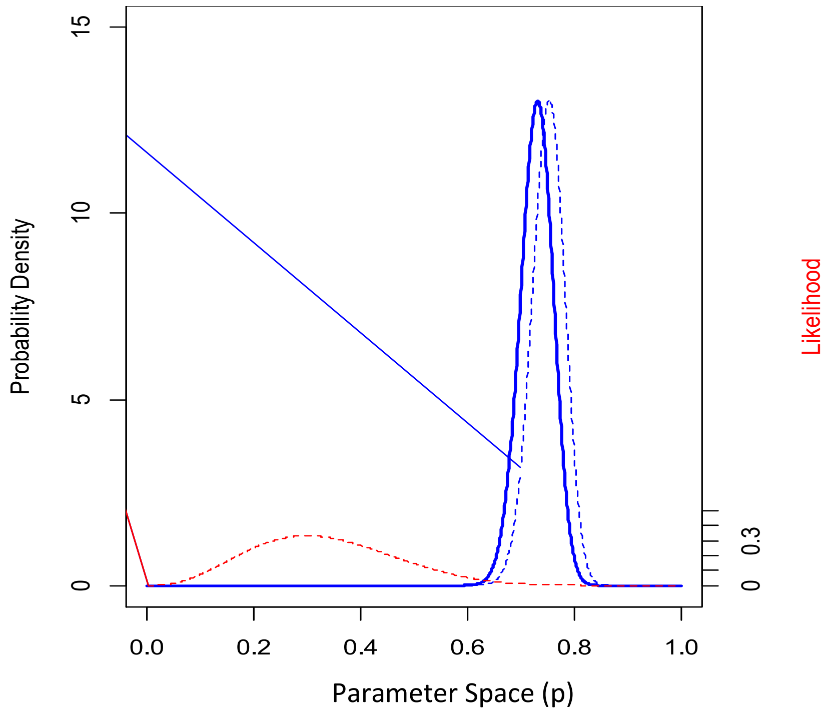

Fig 3: Graph of very informative prior

In the above figure 3, it is interesting to note that likelihood value rises to near 0.3 in red and posterior curve is depicted in dotted blue is bell shaped equally peaked as prior curve in blue. Both are similar and they coincide to great extent.

9.5 Conjugate Prior

Fig 4: Graph of conjugate prior between parameter and probability density

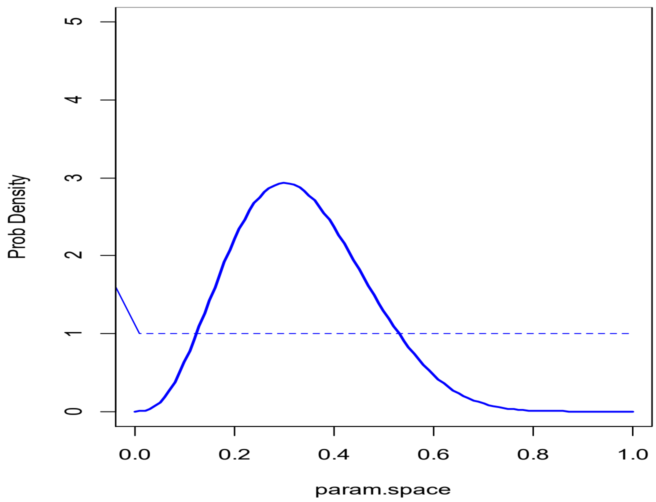

Fig.4 says that, probability density (pdf) remains constant at 1for posterior curve and prior is bell shaped and it pdf is peak at 3 and parameter is around 0.3. Both curves of prior and posterior are quite different.

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9.6 Informative Prior

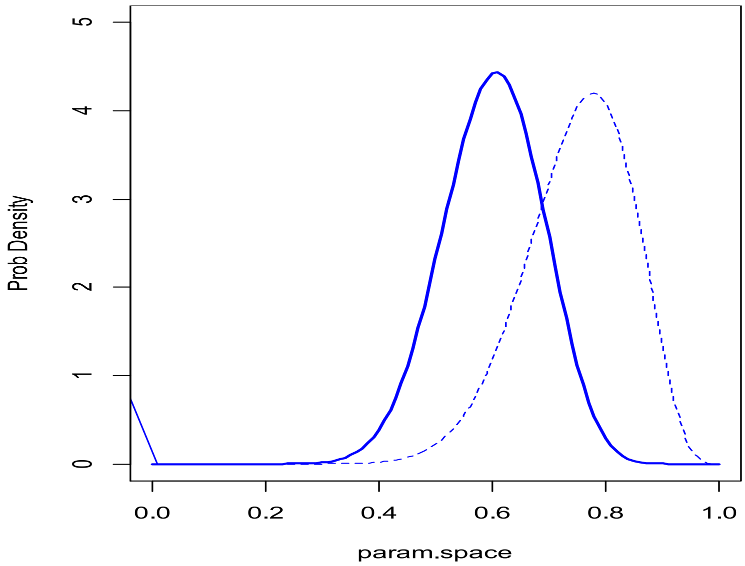

Fig. 5: Graph of informative prior between parameter & probability

Both of the curves of prior and posterior are bell shaped in fig. 5, only point of difference is prior curve is more peaked as compared to posterior curve. The prior curve shows that it's likely to be anywhere between 0 and 1.0, whereas the posterior (shaded blue) shows that it's likely to be anywhere between 0.3 and 1. The fact that the prior curve is more spread out and has a greater peak than the prior curve.

9.7 Super Informative Prior

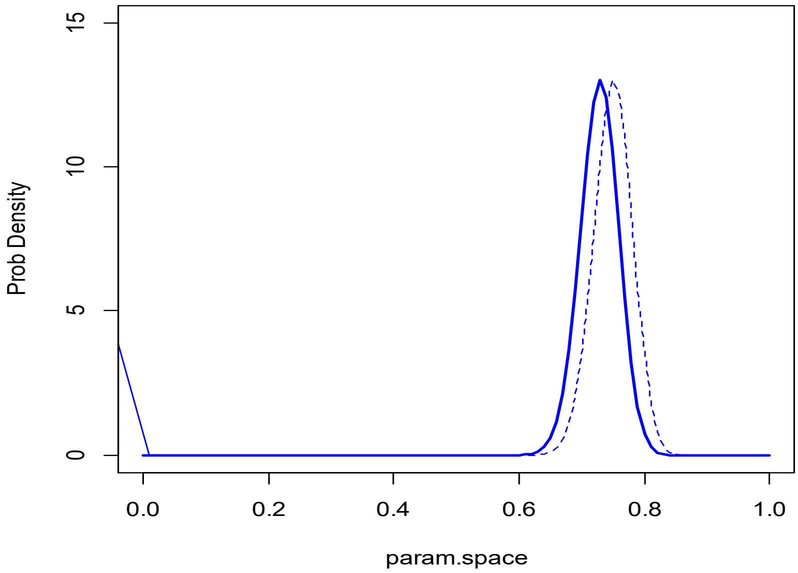

Fig: 6: Graph of Super Informative Prior Between Parameter & Probability

Both of the curves of prior and posterior are bell shaped and almost coincide each other in fig. 6. The prior curve shows that it's likely to be anywhere between 0 and 1.0, whereas the posterior (shaded blue) shows that it's likely to be very limited.

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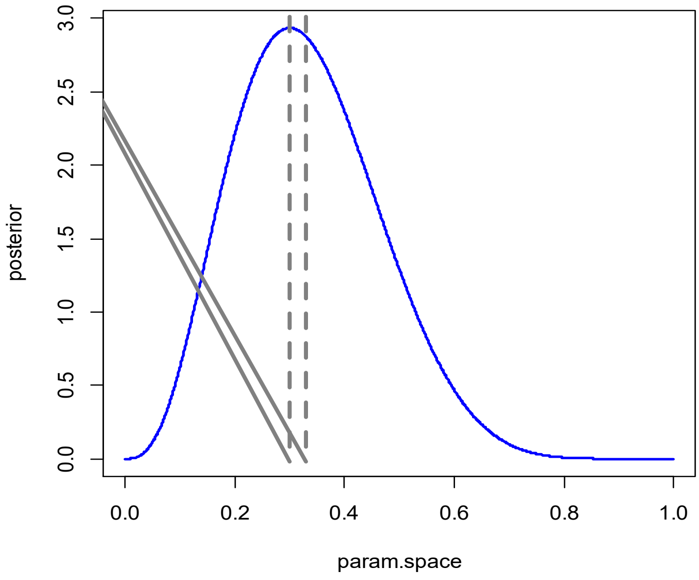

Parameter Space

Fig. 7: Graph of Probability Space and Posterior

This is the graph where posterior tends to provide us its mode as a main result of parameter space with credible interval, 95% credible interval. Mode of posterior is detected for ascertaining the parameter value in the space.

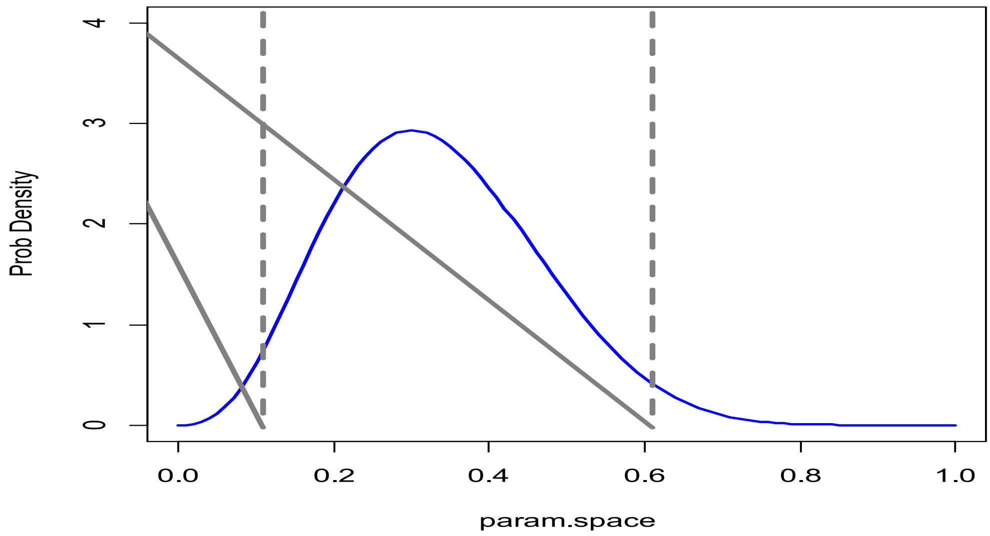

Fig. 8: Bayesian credible interval

Through this figure 8, credible interval has been ascertained in which value of parameter is located and through which HPD is also obtained as an important result.

10. Results and Conclusion

Highest Posterior Density (HPD) interval is computed as [0.11, 0.605] which is the credible interval with 95% confidence.

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Even if we have a CI of 95 percent, Bayesian credible intervals are found to be several. It shows that posterior distribution of 95 percent is enclosed by several intervals. Shortest interval in given all possible intervals is regarded as HPD. It tells us that outside point is less plausible as compared to each point within the interval. In other words, it gives a set of most likely values.

In this way, Bayesian inference analysis has become imperative for obtaining more realistically estimated value of parameter involved in the distribution which was used in the occurrence of idle servers of Markovian queueing models.

Our results derived in the paper are sufficient for the given problem of random occurrence of idle servers of Markovian queueing models because we have used here a conjugate prior that is possible to choose for the random occurrence of idle servers in Markovian queue which is practically found quite realistic by assuming binomial distribution. We have been able to sufficiently deal with this problem of conjugate prior (Beta prior) and posterior of Beta family mathematically as well as computationally to efficiently compute the results by using R algorithm which are easily interpretable as additional information as desired for the model under discussion. However, Bayesian inference gets complicated so as to translate the subjective prior into mathematically developed prior if the condition of conjugacy is not obtained.

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