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Abstract: *In this article, we have considered conformal change of Matsumoto Randers transformation of m^{th} -root metric. Furthermore, we have obtained the condition of projectively flatness, projectively dually flatness with homothetic change and projective factor for said metric. We have also given some fruitful applications of flatness of metric.*

Key words: *Projectively flatness, projectively dually flatness, m^{th} -root metric, Matsumoto Randers Conformal transformation.*

1. Introduction

The investigation of the geometric structures of families of probability distributions has led to the development of information geometry, which finds applications in diverse fields such as multi-terminal information theory, statistical inference, and control systems. Within this framework, Finsler information geometry introduces a distinctive and significant class of Finsler metrics known as dually flat Finsler metrics. These metrics are crucial for understanding flat structures in Finsler information geometry. While studying the information geometry of Riemannian manifolds, Amari and Nagaoka [1] introduced the notion of locally dually flat Riemannian metrics. This concept was later extended to Finsler geometry by Shen [4].

The m^{th} -root metric was first introduced by H. Shimada in 1979 [10]. Since then, both mathematicians and physicists have shown great interest in the study of m^{th} -root Finsler metrics due to their wide range of applications in fields such as ecology, biology, and physics including models of space-time and unified gauge theories. A notable example is the Berwald-Moor metric, a special case of the m^{th} -root metric given by

$$\mathcal{F} = \sqrt[m]{y^{i_1} y^{i_2} y^{i_3} y^{i_4} \dots y^{i_m}},$$

which plays a crucial role in the theory of space-time structure, gravitation, and general relativity.

Let \mathcal{M} be an n -dimensional manifold and $T\mathcal{M}$ be its tangent bundle. Let $\mathcal{F}: T\mathcal{M} \rightarrow \mathfrak{R}$ be defined as $\mathcal{F} = C^{\frac{1}{m}}$, where $C = a_{i_1 i_2 i_3 i_4 \dots i_m}(x) y^{i_1} y^{i_2} y^{i_3} \dots y^{i_m}$ and $a_{i_1 i_2 i_3 i_4 \dots i_m}$ is symmetric in all indices. Then \mathcal{F} is called the m -th root Finsler metric on \mathcal{M} . For $m = 3, 4$, the metric is known as the cubic and quartic metric, respectively. Several researchers have investigated and characterized transformations of m^{th} -root Finsler metrics, including Randers changes, Exponential change and special change [11, 14, 15, 16], conformal Kropina changes [6], and generalized Kropina changes [13]. Tayebi et. al. [11] and M. Kumar [7] have studied generalized m^{th} -root Finsler metrics and explored the conditions under which such metrics are locally dually flat or locally projectively flat.

In this research paper, we consider the conformal Matsumoto-Randers change of a Finsler metric, defined as:

$$\bar{F} = e^{\sigma(x)} \left(\frac{F^2}{F-\beta} + \beta \right) \quad (1)$$

In above metric if $\beta = 0$ and $\sigma_i = 0$, then it is called homothetic change. Hilbert's Fourth Problem, in the regular case, is concerned with characterizing Finsler metrics on an open domain where geodesics are straight lines i.e., that are projectively flat. In Riemannian geometry, it is well known that a Riemannian metric is locally projectively flat if and only if it has constant sectional curvature. Hence, the problem is considered fully resolved in the Riemannian case.

In Finsler geometry, the flag curvature is a natural generalization of sectional curvature. While every locally projectively flat Finsler metric has scalar flag curvature, the converse does not hold there exist Finsler metrics with scalar or even constant flag curvature that are not locally projectively flat [8]. This makes the study of Finsler metrics with scalar or constant flag curvature both natural and necessary, as the problem remains open in Finsler geometry.

In this work, we characterize the conditions under which the conformal Matsumoto-Randers m^{th} -root Finsler metric is locally projectively flat and locally dually flat, and we shall derive meaningful results.

The concept of curvature plays a central role in Riemannian geometry. In 1926, L. Berwald extended the notion of Riemannian curvature to the realm of Finsler geometry, where it continues to be of fundamental importance.

The Riemannian curvature of a Finsler space is defined as a family of linear transformations acting on the tangent space at a point. Specifically, the Riemann curvature $\mathcal{R}_y: T_x\mathcal{M} \rightarrow T_x\mathcal{M}$ is defined by $\mathcal{R}_y(u) = \mathcal{R}_j^i(x, y) \frac{\partial u^j}{\partial u^i}$, $u = \frac{\partial u^j}{\partial u^i}$, where

$$\mathcal{R}_j^i(x, y) = 2 \frac{\partial G^i}{\partial x^j} - y^k \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^k \frac{\partial^2 G^i}{\partial y^k \partial x^j} - \frac{\partial G^i}{\partial x^k} \frac{\partial G^k}{\partial x^j}. \quad (2)$$

The flag curvature K at a point x is a function defined on a tangent plane $P \subset T_x\mathcal{M}$ and a non-zero vector $y \in P$, which measures the curvature of the Finsler space in the direction of y within the plane P . It is denoted by $K = K(x, y, P)$ and defined as follows [4]:

$$K(x, y, P) = \frac{g_y(\mathcal{R}_y(u), u)}{g_y(y, y)g_y(u, u) - [g_y(y, u)]^2}, \quad (3)$$

where $u \in P$ is chosen such that $P = \text{span}\{y, u\}$. If $K(x, y, P) = K(x, y)$, and g_y denotes the fundamental tensor of the Finsler metric at y .

According to Schur's Lemma, in dimension $\dim(\mathcal{M}) \geq 3$, then $K = \text{constant}$, isotropic flag curvature implies constant flag curvature. There exist many non-Riemannian, projectively flat Finsler metrics with constant flag curvature. Examples include the Funk metric and the Hilbert metric [4], which are both projectively flat and have constant flag curvature on strongly convex domains.

These observations motivate the study of constant flag curvature in various classes of Finsler metrics. In this work, we investigate the conformal Matsumoto-Randers change of a Finsler

metric, examining the conditions under which such a metric reduces constant or scalar flag curvature.

2. Projectively flatness with conformal change of Matsumoto Randers transformation of m^{th} -root metric

A Finsler metric $\mathcal{F} = \mathcal{F}(x, y)$, defined on an open subset $\mathbb{V} \subset \mathfrak{R}^n$, is projectively flat if and only if it satisfies $[\overline{\mathcal{F}}]_{x^k y^l} y^k = [\overline{\mathcal{F}}]_{x^l}$. For m^{th} -root metric $\mathcal{F} = C^{\frac{1}{m}}$, we have used the following notations

$$C_i = \frac{\partial C}{\partial y^i}, \quad C_{x^i} = \frac{\partial C}{\partial x^i}, \quad C_{0l} = C_{x^k y^l} y^k = \frac{\partial^2 C}{\partial x^k y^l} y^k.$$

Differentiating equation (1) with respect to x^i , we obtain

$$\begin{aligned} \overline{F}_{x^i} = & \frac{e^\sigma}{m \left(C^{\frac{1}{m} - \beta} \right)^2} \left[C^{\frac{3}{m} - 1} C_{x^i} + 2m C^{\frac{2}{m}} \beta_{x^i} - 2C^{\left(\frac{2}{m} - 1 \right)} C_{x^i} \beta - 2m C^{\frac{1}{m}} \beta \beta_{x^i} - 3m \beta^2 \beta_{x^i} + \right. \\ & \left. m C^{\frac{3}{m}} \sigma_{x^i} - 2m C^{\frac{1}{m}} \beta^2 \sigma_{x^i} + m \beta^2 \sigma_{x^i} \right], \end{aligned} \quad (2)$$

Differentiating equation (2) with respect to y^l and then multiplying the result by y^i yields

$$\begin{aligned} \overline{F}_{x^i y^l} y^i = & \frac{e^\sigma}{m \left(C^{\frac{1}{m} - \beta} \right)^3} \left[\left(\frac{3}{m} - 1 \right) C^{\frac{4}{m} - 2} C_l C_0 - \frac{3}{m} C^{\frac{3}{m} - 2} \beta_{cl} C_0 + C^{\frac{4}{m} - 1} C_0 C_l - \beta C^{\frac{3}{m} - 1} C_0 C_l + \right. \\ & 4C^{\frac{3}{m} - 1} C_l \beta_0 - 4\beta C^{\frac{2}{m} - 5} C_l \beta_0 + 2m C^{\frac{3}{m}} \beta_{0l} - 2m \beta C^{\frac{2}{m}} \beta_{0l} - 2 \left(\frac{2}{m} - 1 \right) C^{\frac{3}{m} - 2} C_l C_0 \beta + 2 \left(\frac{2}{m} - \right. \\ & \left. 1 \right) C^{\frac{2}{m} - 2} C_l C_0 \beta^2 - 2C^{\frac{3}{m} - 1} C_{0l} \beta + 2C^{\frac{2}{m} - 1} \beta^2 C_{0l} - 2C^{\frac{3}{m} - 1} C_0 \beta_l + 2C^{\frac{2}{m} - 1} C_0 \beta_l \beta_1 - \\ & 2C^{\frac{2}{m} - 1} C_l \beta \beta_0 + 2C^{\frac{1}{m} - 1} C_l \beta^2 \beta_0 - 2m C^{\frac{2}{m}} \beta_l \beta_0 + 2m C^{\frac{1}{m}} \beta \beta_l \beta_0 - 2m C^{\frac{2}{m}} \beta \beta_{0l} - \\ & 2m C^{\frac{2}{m}} \beta \beta_{0l} + 2m C^{\frac{1}{m}} \beta^2 \beta_{0l} - 6m \beta C^{\frac{1}{m}} \beta_l \beta_0 + 6m \beta^2 \beta_l \beta_0 - 3m \beta^2 C^{\frac{1}{m}} \beta_{0l} + 3m \beta^3 \beta_{0l} + \\ & 3C^{\frac{4}{m} - 1} C_l \sigma_0 - 3C^{\frac{3}{m} - 1} \beta C_l \sigma_0 - 2C^{\frac{2}{m} - 1} C_l \beta^2 \sigma_0 + 2\beta^3 C^{\frac{1}{m} - 1} C_l \sigma_0 - 4m C^{\frac{2}{m}} \beta \beta_l \sigma_0 + \\ & 4m \beta^2 C^{\frac{1}{m}} \beta_l \sigma_0 + 2m \beta C^{\frac{1}{m}} \beta_l \sigma_0 - 2m \beta^2 \beta_l \sigma_0 + \frac{2}{m} C^{\frac{4}{m} - 2} C_0 C_l - 2\beta_l C^{\frac{3}{m} - 1} C_0 + 4C^{\frac{3}{m}} \beta_0 C_l - \\ & 4m C^{\frac{2}{m}} \beta_0 \beta_l - \frac{4}{m} C^{\frac{3}{m} - 2} C_l C_0 \beta + 4\beta_l \beta C_l C_0 C^{\frac{2}{m} - 1} + 4C_0 \beta \beta_l C^{\frac{2}{m} - 1} - 4C^{\frac{2}{m} - 1} \beta \beta_0 C_l + \\ & 4m \beta_l \beta_0 \beta C_l C^{\frac{1}{m}} - 6\beta^2 \beta_0 C^{\frac{1}{m} - 1} C_l + 6m \beta_l \beta^2 \beta_0 + 2C^{\frac{4}{m} - 1} \sigma_0 C_l - 2\beta \beta_l C^{\frac{3}{m}} \sigma_0 - 4C^{\frac{2}{m} - 1} C_l \beta^2 \sigma_0 + \\ & \left. 4m \beta_l C^{\frac{1}{m}} \beta^2 \sigma_0 - 2C^{\frac{1}{m} - 1} C_l \beta^2 \sigma_0 - 2m \beta^2 \sigma_0 \beta_l \right]. \end{aligned} \quad (3)$$

Now, if we consider \overline{F} to be a projectively flat Finsler metric, then it must satisfy the condition $[\overline{F}]_{x^i y^l} y^i - [\overline{F}]_{x^l} = 0$. In view of equations (2) and (3), this condition becomes

$$\begin{aligned}
 & C^{\frac{4}{m}-2} \frac{2}{m} C_0 C_l + \left(\frac{3}{m} - 1\right) C^{\frac{4}{m}-2} C_0 C_l - C^{\frac{3}{m}-2} \left(\frac{3}{m} \beta C_l C_0 + 2 \left(\frac{2}{m} - 1\right) C_0 C_l \beta + \frac{4}{m} C_l C_0 \beta\right) + \\
 & C^{\frac{3}{m}} (2m\beta_{0l} + 4\beta_0 C_l - 2\beta_0 \beta_l \sigma_0 - 2m\beta_{nl} + m\beta \sigma_{xl}) - C^{\frac{2}{m}} (2m\beta\beta_{0l} + 2m\beta_0 \beta_l + 2m\beta\beta_{0l} + \\
 & 4m\beta\beta_l \sigma_0 + 4m\beta_0 \beta_l - 4m\beta\beta_{xl} - 2m\beta^2 \sigma_{xl}) + C^{\frac{4}{m}-1} (C_0 C_l + 5C_l \sigma_0 - C_{xl}) - \\
 & C^{\frac{3}{m}-1} (\beta C_0 C_l - 4C_l \beta_0 + 2C_{0l} \beta + 2C_0 \beta_l + 3\beta C_l \sigma_0 - \beta C_{xl} + 2\beta_l C_0 - 2C_{xl} \beta) - \\
 & C^{\frac{2}{m}-1} (4\beta C_l \beta_0 - 2\beta^2 C_{0l} - 2C_0 \beta_l \beta + 2C_l \beta \beta_0 + 2C_l \beta^2 \sigma_0 - 4\beta_l \beta C_0 - 4C_0 \beta \beta_l + 4\beta \beta_0 C_l + \\
 & 4C_l \beta^2 \sigma_0 - 6\beta^2 \beta_0 C C_l + 2C_{xl} \beta^2) + C^{\frac{2}{m}-2} (2\left(\frac{2}{m} - 1\right) C_l C_0 \beta^2) + C^{\frac{1}{m}-1} (2C_l \beta^2 \beta_0 + 2\beta^3 C_l \sigma_0 - \\
 & 2C_l \beta^2 \sigma_0) + C^{\frac{1}{m}} (2m\beta\beta_l \beta_0 + 2m\beta^2 \beta_{0l} - 6m\beta\beta_l \beta_0 - 3m\beta^2 \beta_{0l} + 4m\beta^2 \beta_l \sigma_0 + 2m\beta\beta_l \sigma_0 + \\
 & 4m\beta_l \beta_0 \beta C_l + 4m\beta_l \beta^2 \sigma_0 - 2m\beta^2 \beta_{xl} + 3m\beta^2 \beta_{xl} - 2m\beta^3 \sigma_{xl} - m\beta^2 \sigma_{xl}) + C^{\frac{4}{m}} (-m\sigma_{xl}) + \\
 & (6m\beta^2 \beta_l \beta_0 + 3m\beta^3 \beta_{0l} - 2m\beta^2 \beta_l \sigma_0 + 6m\beta_l \beta^2 \beta_0 - 2m\beta^2 \sigma_0 \beta_l - 3m\beta^3 \beta_{nl} + m\beta^3 \sigma_{nl}) = \\
 & 0
 \end{aligned} \tag{4}$$

To evaluate the above equation, we proceed by applying the relevant lemma.

Lemma 2.1: [12] Let $\mathcal{F} = C^{\frac{1}{m}}$ ($m > 2$), be an m^{th} -root Finsler metric on an open subset $\mathbb{V} \subset \mathbb{R}^n$. Suppose that the equation $PC^{\frac{5}{m}-r} + QC^{\frac{4}{m}-r} + RC^{\frac{3}{m}-r} + SC^{\frac{2}{m}-r} + TC^{\frac{1}{m}-r} + U = 0$ where $r = 0, 1, 2, 3, 4, 5, \dots$ and P, Q, R, S, T and U are homogeneous polynomials in y . Then $P = Q = R = S = T = U = 0$.

Using lemma (2.1), the equation (4) reduces to

$$\frac{2}{m} C_0 C_l + \left(\frac{3}{m} - 1\right) C_0 C_l = 0, \tag{5}$$

$$\frac{3}{m} \beta C_l C_0 + 2 \left(\frac{2}{m} - 1\right) C_0 C_l \beta + \frac{4}{m} C_l C_0 \beta = 0, \tag{6}$$

$$2m\beta_{0l} + 4\beta_0 C_l - 2\beta b_l \sigma_0 - 2m\beta_{xl} + m\beta \sigma_{xl} = 0, \tag{7}$$

$$2m\beta\beta_{0l} + 2m\beta_0 b_l + 2m\beta\beta_{0l} + 4m\beta b_l \sigma_0 + 4m\beta_0 b_l - 4m\beta\beta_{nl} - 2m\beta^2 \sigma_{xl} = 0, \tag{8}$$

$$C_0 C_l + 5C_l \sigma_0 - C_{xl} = 0, \tag{9}$$

$$\beta C_0 C_l - 4C_l \beta_0 + 2C_{0l} \beta + 2C_0 b_l + 3\beta C_l \sigma_0 - \beta C_{xl} + 2b_l C_0 - 2C_{xl} \beta = 0, \tag{10}$$

$$4\beta C_l \beta_0 - 2\beta^2 C_{0l} - 2C_0 b_l \beta + 2C_l \beta \beta_0 + 2C_l \beta^2 \sigma_0 - 4b_l \beta C_0 - 4C_0 \beta b_l + 4\beta \beta_0 C_l + \\
 4C_l \beta^2 \sigma_0 + 2C_{xl} \beta^2 = 0, \tag{11}$$

$$2\left(\frac{2}{m} - 1\right) C_l C_0 \beta^2 = 0, \tag{12}$$

$$2C_l \beta^2 \beta_0 + 2\beta^3 C_l \sigma_0 - 6\beta^2 \beta_0 C_l - 2C_l \beta^2 \sigma_0 = 0, \tag{13}$$

$$2m\beta b_l \beta_0 + 2m\beta^2 \beta_{0l} - 6m\beta b_l \beta_0 - 3m\beta^2 \beta_{0l} + 4m\beta^2 b_l \sigma_0 + 2m\beta b_l \sigma_0 + 4mb_l \beta_0 \beta C_l + \\
 4mb_l \beta^2 \sigma_0 - 2m\beta^2 \beta_{xl} + 3m\beta^2 \beta_{xl} - \beta 2m\beta^3 \sigma_{xl} - m\beta^2 \sigma_{xl} = 0, \tag{14}$$

$$m\sigma_{xl} = 0, \tag{15}$$

$$6m\beta^2 b_l \beta_0 + 3m\beta^3 \beta_{0l} - 2m\beta^2 b_l \sigma_0 + 6mb_l \beta^2 \beta_0 - 2m\beta^2 \sigma_0 b_l - 3m\beta^3 \beta_{xl} + m\beta^3 \sigma_{xl} = \\
 0, \tag{16}$$

By (10), (11), (13) and (14) we get

$$b_l C_0 + \beta_0 C_l = 0 \tag{17}$$

In view of equation (12) we have $C_0 = 0$ as $C_l \neq 0$. From equation (17) and $C_0 = 0$ it follows that $\beta_0 = 0$. Taking partial derivative of $\beta_0 = 0$, we get

$$\beta_{0l} + \beta_{x^l} = 0. \tag{18}$$

From equation (15), we have $\sigma_{x^j} = 0$ i.e. $\sigma_0 = 0$. Again from equation (16) and $\sigma_0 = 0$, we have obtain $\beta_{0l} + \beta_{x^l} = 0$. By considering (18), we get $\beta_{x^l} = 0$, which implies $b_j = \text{constant}$. Then we have

Theorem: Let $\mathcal{F} = C^{\frac{1}{m}}$ be m^{th} -root Finsler metric and $\bar{\mathcal{F}} = \bar{\mathcal{F}}(x, y)$ be the conformal Matsumoto Randers change of \mathcal{F} . Then $\bar{\mathcal{F}}$ to be projectively flat if and only if $\beta_{x^l} = 0$, $b_i = \text{constant}$ and the change is homothetic.

3. Locally Dually flatness with conformal change of Matsumoto Randers transformation of m^{th} -root metric

A Finsler metric $\mathcal{F} = \mathcal{F}(x, y)$ defined on an open subset $\mathbb{U} \in \mathbb{R}^n$, is said to be locally dually flat if and only if it satisfies the following condition [4]:

$$[\bar{\mathcal{F}}^2]_{x^i y^l} y^i - 2[\bar{\mathcal{F}}^2]_{x^l} = 0$$

Using equation (1), we obtain:

$$[\bar{\mathcal{F}}^2]_{x^i} = \frac{2e^{2\sigma}}{\left(\frac{1}{C^{\frac{1}{m}-\beta}}\right)^3} \left[\frac{1}{m} C^{\frac{5}{m}-1} C_i - \frac{1}{m} C^{\frac{4}{m}-1} \beta C_i - \frac{3}{m} C^{\frac{3}{m}-1} C_i \beta^2 + \frac{2}{m} C^{\frac{2}{m}-1} \beta^3 C_i - 3\beta^2 \sigma_i C^{\frac{3}{m}} - C^{\frac{2}{m}} (3\beta^2 \beta_i - \beta^3 \sigma_i) + C^{\frac{4}{m}} (2\beta_i + \beta \sigma_i) + C^{\frac{5}{m}} \sigma_i - \beta^4 \beta_i - \beta^5 \sigma_i \right] \tag{19}$$

Differentiating equation (19) with respect to y^l and then contracting the result with y^l , we obtain

$$[\bar{\mathcal{F}}^2]_{x^i y^l} y^l = \frac{2e^{2\sigma}}{\left(\frac{1}{C^{\frac{1}{m}-\beta}}\right)^4} \left[\left(-\frac{1}{m^2} - 1 \right) C^{\frac{6}{m}-2} C_0 C_l + \frac{1}{m} C^{\frac{5}{m}-1} (5C_l \sigma_0 + C_{0l}) \pm \left(\frac{11}{m^2} - \frac{1}{m} \right) C^{\frac{5}{m}-2} \beta C_0 C_l - C^{\frac{5}{m}-1} \left(\frac{2\beta}{m} C_{0l} + \frac{7}{m} \beta_l C_0 + \frac{20}{m} C_l \beta_0 + \frac{11}{m} \beta C_l \sigma_0 + \frac{1}{m} \left(\frac{4}{m} - 1 \right) \beta C_0 C_l \right) \right] - C^{\frac{4}{m}-2} \left\{ \left(\frac{31}{m^2} - \frac{4}{m} \right) C_0 C_l \beta^2 \right\} S - C^{\frac{4}{m}-1} \left\{ \frac{11}{m} \beta \beta_l C_0 + \frac{2}{m} C_{0l} + \left(9 + \frac{9}{8} \right) \beta^2 \sigma_0 C_l + \frac{8}{m} \beta_0 \beta C_l \right\} + C^{\frac{3}{m}-2} \left\{ \left(\frac{1}{m^2} - \frac{5}{m} \right) \beta_0 C_0 C_l \right\} + C^{\frac{3}{m}-1} \left\{ \frac{5}{m} C_{0l} - \frac{6}{m} C_0 \beta^2 \beta_l + \left(9 - \frac{9}{4} \right) \beta^3 \sigma_0 C_l + \frac{12}{m} \beta^2 \beta_0 C_l \right\} - C^{\frac{2}{m}-2} \left\{ \frac{2}{m} \left(\frac{2}{m} - 1 \right) \beta^4 C_0 C_l \right\} - C^{\frac{2}{m}-1} \left\{ \frac{2}{m} C_{0l} \beta^4 + \frac{3}{m} \beta^3 \beta_0 C_l + \frac{17}{m} \beta^4 \sigma_0 C_l - \frac{12}{m} \beta_l \beta^3 C_0 \right\} - C^{\frac{4}{m}} \{ \sigma_0 \beta_l + 2\beta \beta_{0l} - 12\beta_0 \beta_l \} + C^{\frac{3}{m}} \{ 6\sigma_0 \beta_l - 9\beta^2 \sigma_0 \beta_l - 6\beta \beta_0 \beta_l - 9\beta^2 \beta_{0l} \} + C^{\frac{1}{m}-1} \left\{ \frac{5}{m} C_l \beta^4 \beta_0 + \frac{3}{m} \beta^5 \sigma_0 C_l \right\} + C^{\frac{2}{m}} \{ 6\beta^3 \beta_{0l} - 3 \} + \{ \beta^5 \beta_{0l} - 2\beta^4 \beta_0 \beta_l - \beta^3 \beta_l \sigma_0 \} \tag{20}$$

If $\bar{\mathcal{F}}$ is a locally dually flat Finsler metric, then it satisfies the condition

$$[\bar{\mathcal{F}}^2]_{x^k y^l} y^k - 2[\bar{\mathcal{F}}^2]_{x^l} = 0$$

From this equation, we obtain

$$\begin{aligned}
 & C_m^{5-2} \left(\frac{11}{m^2} - \frac{1}{m} \right) \beta C_0 C_l - C_m^{5-1} \left\{ \left(\frac{7}{m} \beta_l C_0 + \frac{20}{m} C_l \beta_0 + \frac{11}{m} \beta C_l \sigma_0 + \frac{1}{m} \left(\frac{4}{m} - 1 \right) \beta C_0 C_l - \frac{2\beta}{m} C_l - \right. \right. \\
 & \left. \left. \frac{2}{m} \beta C_l \right) \right\} - C_m^{4-2} \left\{ \left(\frac{31}{m^2} - \frac{4}{m} \right) C_0 C_l \beta^2 \right\} - C_m^{4-1} \left\{ \frac{11}{m} \beta \beta_l C_0 + \frac{2}{m} C_{0l} + \left(9 + \frac{9}{8} \right) \beta^2 \sigma_0 C_l + \right. \\
 & \left. \frac{8}{m} \beta_0 \beta C_l + \frac{2}{m} \beta^2 C_l - \frac{6}{m} C_l \beta^2 \right\} + C_m^{3-2} \left\{ \left(\frac{1}{m^2} - \frac{5}{m} \right) \beta_0 C_0 C_l \right\} + C_m^{3-1} \left\{ \frac{5}{m} C_{0l} - \frac{6}{m} C_0 \beta^2 \beta_l + \right. \\
 & \left(9 - \frac{9}{4} \right) \beta^3 \sigma_0 C_l + \frac{12}{m} \beta^2 \beta_0 C_l - \frac{6}{m} C_l \beta^3 - \frac{4}{m} \beta^3 C_l \left. \right\} - C_m^{2-2} \left\{ \frac{2}{m} \left(\frac{2}{m} - 1 \right) \beta^4 C_0 C_l \right\} - \\
 & C_m^{2-1} \left\{ \frac{2}{m} C_{0l} \beta^4 + \frac{3}{m} \beta^3 \beta_0 C_l + \frac{17}{m} \beta^4 \sigma_0 C_l - \frac{12}{m} \beta_l \beta^3 C_0 + \frac{4}{m} \beta^4 C_l \right\} - C_m^4 \{ \sigma_0 \beta_l + 2\beta \beta_{0l} - \\
 & 12\beta_0 \beta_l - 6\beta^2 \sigma_0 + (4\beta \beta_l + 2\beta^2 \sigma_l) \} + C_m^3 \{ 6\sigma_0 \beta_l - 9\beta^2 \sigma_0 \beta_l - 6\beta \beta_0 \beta_l - 9\beta^2 \beta_{0l} - \\
 & 6\beta^3 \sigma_l + (6\beta^2 \beta_l - 2\beta^3 \sigma_l) \} + C_m^{1-1} \left\{ \frac{5}{m} C_l \beta^4 \beta_0 + \frac{3}{m} \beta^5 \sigma_0 C_l \right\} C_m^2 \{ 6\beta^2 \beta_{0l} - 3\beta_0 \beta_l \beta^2 + \\
 & 15\beta^3 \sigma_0 \beta_l - (6\beta^3 \beta_l - 2\beta^4 \sigma_l) \} + C_m^1 \{ 5\beta^3 \beta_0 \beta_l - 4\beta^4 \beta_{0l} + \beta^4 \sigma_0 \beta_l + (2\beta^4 \beta_l + \\
 & 2\beta^5 \sigma_l) \} + C_m^5 \{ 2\beta_{0l} + \beta_l \sigma_0 (4\beta_l 2\beta \sigma_l) + -2\sigma_l \beta \} - 2\sigma_l C_m^6 + \{ \beta^5 \beta_{0l} - 2\beta^4 \beta_0 \beta_l - \beta^3 \beta_l \sigma_0 - \\
 & 2\beta^5 \beta_l - 2\beta^6 \sigma_l \} = 0 \tag{21}
 \end{aligned}$$

To evaluate the above equation, we proceed by applying the relevant lemma.

Lemma 3.1: [12] Let $\mathcal{F} = C_m^{\frac{1}{m}}$ ($m > 2$), be an m^{th} -root Finsler metric on an open subset $\mathbb{V} \subset \mathbb{R}^n$. Suppose that the equation $PC_m^{\frac{5}{m}-r} + QC_m^{\frac{4}{m}-r} + RC_m^{\frac{3}{m}-r} + SC_m^{\frac{2}{m}-r} + TC_m^{\frac{1}{m}-r} + U = 0$ where $r = 0, 1, 2, 3, 4, 5, \dots$ and P, Q, R, S, T and U are homogeneous polynomials in y . Then $P = Q = R = S = T = U = 0$.

Using lemma (3.1) in equation (21), it reduces to

$$\beta \left(\frac{1}{m^2} - \frac{1}{m} \right) C_0 C_l = 0, \tag{22}$$

$$\frac{1}{m} (5C_l \sigma_0 + C_{0l} - 2C_l) = 0, \tag{23}$$

$$\left(\frac{11}{m^2} - \frac{1}{m} \right) \beta C_0 C_l = 0, \tag{24}$$

$$\frac{2\beta}{m} C_{0l} + \frac{7}{m} \beta_l C_0 + \frac{20}{m} C_l \beta_0 + \frac{11}{m} \beta C_l \sigma_0 + \frac{1}{m} \left(\frac{4}{m} - 1 \right) \beta C_0 C_l - \frac{2\beta}{m} C_l - \frac{2}{m} \beta C_l = 0, \tag{25}$$

$$\left(\frac{31}{m^2} - \frac{4}{m} \right) C_0 C_l \beta^2 = 0, \tag{26}$$

$$\frac{11}{m} \beta \beta_l C_0 + \frac{2}{m} C_{0l} + \left(9 + \frac{8}{m} \right) \beta^2 \sigma_0 C_l + \frac{8}{m} \beta_0 \beta C_l + \frac{2}{m} \beta^2 C_l - \frac{6}{m} C_l \beta^2 = 0, \tag{27}$$

$$\left(\frac{1}{m^2} - \frac{5}{m} \right) \beta_0 C_0 C_l = 0, \tag{28}$$

$$\frac{5}{m} C_{0l} - \frac{6}{m} C_0 \beta^2 \beta_l + \left(9 - \frac{4}{m} \right) \beta^3 \sigma_0 C_l + \frac{12}{m} \beta^2 \beta_0 C_l - \frac{6}{m} C_l \beta^3 - \frac{4}{m} \beta^3 C_l = 0, \tag{29}$$

$$\frac{2}{m} \left(\frac{2}{m} - 1 \right) \beta^4 C_0 C_l = 0, \tag{30}$$

$$\frac{2}{m} C_{0l} \beta^4 + \frac{3}{m} \beta^3 \beta_0 C_l + \frac{17}{m} \beta^4 \sigma_0 C_l - \frac{12}{m} \beta_l \beta^3 C_0 + \frac{4}{m} \beta^4 C_l = 0, \tag{31}$$

$$\sigma_0 \beta_l + 2\beta \beta_l - 12\beta_0 \beta_l - 6\beta^2 \sigma_0 + (4\beta \beta_l + 2\beta^2 \sigma_l) = 0, \tag{32}$$

$$6\sigma_0\beta_l - 9\beta^2\sigma_0\beta_l - 6\beta\beta_0\beta_l - 9\beta^2\beta_{0l} - 6\beta^3\sigma_l + (6\beta^2\beta_l - 2\beta^3\sigma_l)=0, \quad (33)$$

$$\frac{5}{m}C_l\beta^4\beta_0 + \frac{3}{m}\beta^5\sigma_0C_l=0, \quad (34)$$

$$6\beta^2\beta_{0l} - 3\beta_0\beta_l\beta^2 + 15\beta^3\sigma_0\beta_l - (6\beta^3\beta_l - 2\beta^4\sigma_l)=0, \quad (35)$$

$$5\beta^3\beta_0\beta_l - 4\beta^4\beta_{0l} + \beta^4\sigma_0\beta_l + (2\beta^4\beta_l + 2\beta^5\sigma_l)=0, \quad (36)$$

$$2\beta_{0l} + \beta_l\sigma_0 - (4\beta_l + 2\beta\sigma_l) + 2\sigma_l\beta = 0, \quad (37)$$

$$2\sigma_{x^l} = 0, \quad (38)$$

$$\beta^5\beta_{0l} - 2\beta^4\beta_0\beta_l - \beta^3\beta_l\sigma_0 - \beta^5\beta_{x^l} = 0. \quad (39)$$

Since $C_l \neq 0$, therefore from equations (22), (24), (26) and (30) gives $C_0 = 0$, Then from equation (28) we get $\beta_0 = 0$. Taking partial derivative of $\beta_0 = 0$, we have obtained

$$\beta_{0l} + \beta_{x^l} = 0. \quad (40)$$

From equations (22), (23), (24) and (25), we obtain $\sigma_0 = 0$. Again from equation (39) and using $\sigma_0 = 0$, we have $\beta_{0l} + \beta_{x^l} = 0$. By considering (40), we get $\beta_{x^l} = 0$, which implies $b_j = \text{constant}$. Then we have

Theorem: Let $\mathcal{F} = C^{\frac{1}{m}}$ be m^{th} -root Finsler metric and $\bar{\mathcal{F}} = \bar{\mathcal{F}}(x, y)$ be the conformal Matsumoto Randers change of \mathcal{F} . Then $\bar{\mathcal{F}}$ to be locally dually flat if and only if $\beta_{x^l} = 0$, $b_i = \text{constant}$ and the change is homothetic.

4. Constant flag Curvature in conformal change of Matsumoto Randers transformation of m^{th} -root metric

In this section, we have derived the condition under which the transformed Finsler metric is projectively flat with constant flag curvature. The scalar flag curvature K of a projectively flat Finsler metric F is given by $K = F^{-2}(P^2 - P_{x^i}y^i)$, where P is the projective factor, defined as [4]

$$P = \frac{\mathcal{F}_{x^i}y^i}{2\mathcal{F}}, \quad (41)$$

From equation (1), we have

$$\begin{aligned} [\bar{\mathcal{F}}_{x^l}]y^j &= \frac{e^\sigma}{m\left(C^{\frac{1}{m}-\beta}\right)^2} \left[C^{\frac{3}{m}-1}C_0 + 2mC^{\frac{2}{m}}\beta_0 - 2C^{\frac{2}{m}-1}C_0\beta - 2mC^{\frac{1}{m}}\beta\beta_0 - 3m\beta^2\beta_0 + \right. \\ &\left. mC^{\frac{3}{m}}\sigma_0 - 2mC^{\frac{1}{m}}\beta^2\sigma_0 + m\beta^2\sigma_0 \right], \end{aligned} \quad (42)$$

Irreducibility of C and $\text{deg}(C_{x^i}) = m - 1$ gives us that there exists a one form $\theta = \theta_i(x)y^i$ such that $C_0 = m\theta C$. Then above equation can be written as

$$\begin{aligned} [\bar{\mathcal{F}}_{x^l}]y^j &= \frac{e^\sigma}{m\left(C^{\frac{1}{m}-\beta}\right)^3} \left[m\theta C^{\frac{3}{m}} + 2mC^{\frac{2}{m}}\beta_0 - 2C^{\frac{2}{m}}m\theta\beta - 2mC^{\frac{1}{m}}\beta\beta_0 - 3m\beta^2\beta_0 + mC^{\frac{3}{m}}\sigma_0 - \right. \\ &\left. 2mC^{\frac{1}{m}}\beta^2\sigma_0 + m\beta^2\sigma_0 \right]. \end{aligned} \quad (43)$$

By considering equation (41), the projective factor of the transformed Finsler metric \bar{F} is given by

$$\bar{P} = \frac{1}{2\left(\frac{1}{C^m}-\beta\right)} \left[m\theta C^{\frac{3}{m}} + 2mC^{\frac{2}{m}}\beta_0 - 2C^{\frac{2}{m}}m\theta\beta - 2mC^{\frac{1}{m}}\beta\beta_0 - 3m\beta^2\beta_0 + mC^{\frac{3}{m}}\sigma_0 - 2mC^{\frac{1}{m}}\beta^2\sigma_0 + m\beta^2\sigma_0 \right]. \quad (44)$$

Now, differentiating above equation with respect to x^i , we have obtained

$$\begin{aligned} \bar{P}_{x^i} = & \frac{1}{2\left(\frac{1}{C^m}-\beta\right)^2} \left[m\theta_{x^i}C^{\frac{4}{m}} + 3C^{\frac{4}{m}-1}C_{x^i}\theta + 4C^{\frac{3}{m}-1}C_{x^i}\beta_0 + 2mC^{\frac{3}{m}}\beta_{0x^i} - 4C^{\frac{3}{m}-1}C_{x^i}m\beta\theta - \right. \\ & 2C^{\frac{3}{m}}\theta_{x^i}\beta - 2m\theta C^{\frac{2}{m}}\beta_{x^i} - 2C^{\frac{2}{m}-1}C_{x^i}\beta_0\beta - 2mC^{\frac{2}{m}}\beta_{x^i}\beta_0 - 2mC^{\frac{3}{m}}\beta_{0x^i}\beta - 3mC^{\frac{1}{m}}\beta_{0x^i}\beta^2 - \\ & 6m\beta C^{\frac{1}{m}}\beta_{x^i}\beta_0 + mC^{\frac{4}{m}}\sigma_{0x^i} + 3C^{\frac{4}{m}-1}C_{x^i}\sigma_0 - 2mC^{\frac{2}{m}}\sigma_{0x^i}\beta^2 - 2C^{\frac{2}{m}-1}C_{x^i}\sigma_0\beta^2 - \\ & 4m\beta C^{\frac{2}{m}}\beta_{x^i}\sigma_0 + 2m\beta C^{\frac{1}{m}}\beta_{x^i}\sigma_0 + mC^{\frac{1}{m}}\sigma_{0x^i}\beta^2 - m\beta\theta_{x^i}C^{\frac{3}{m}} - 3C^{\frac{3}{m}-1}C_{x^i}\beta\theta - 4C^{\frac{2}{m}}\beta_0\beta - \\ & 2m\beta C^{\frac{2}{m}}\beta_{0x^i} + 4m\theta\beta^2 C^{\frac{2}{m}-1}\beta_{x^i} + 2m\beta^2\theta_{x^i}C^{\frac{2}{m}} + 2\beta m\theta\beta_{x^i}C^{\frac{1}{m}} + 2\beta^2\beta_0 C_{x^i}C^{\frac{1}{m}-1} + \\ & 2m\beta C^{\frac{1}{m}}\beta_{x^i}\beta_0 + 2m\beta^2\beta_{0x^i}C^{\frac{1}{m}} + 3m\beta^3\beta_{0x^i} + 6m\beta^2\beta_{x^i}\beta_0 - m\beta C^{\frac{3}{m}}\sigma_{0x^i} - 3\beta C^{\frac{3}{m}-1}C_{x^i}\sigma_0 + \\ & 3m\beta_{0x^i}\beta^3 + 6m\beta^2\beta_{0x^i}\beta_0 - m\beta\beta_{0x^i}C^{\frac{3}{m}} - 3\beta C^{\frac{3}{m}-1}C_{x^i}\sigma_0 + 2m\beta^3 C^{\frac{1}{m}}\sigma_{0x^i} + \\ & 2m\beta^3 C^{\frac{1}{m}-1}\sigma_0 C_{x^i} + 4m\beta^2 C^{\frac{1}{m}}\sigma_0\beta_{x^i} - 2m\beta^2\sigma_0\beta_{x^i} - m\beta^3\sigma_{0x^i} - \theta C^{\frac{4}{m}-1}C_{x^i} + m\theta C^{\frac{3}{m}}\beta_{x^i} - \\ & 2C^{\frac{1}{m}-1}\beta_0 C_{x^i} + 2mC^{\frac{1}{m}}\beta_0\beta_{x^i} + 2\theta\beta C^{\frac{3}{m}-1}C_{x^i} - 2m\beta\theta C^{\frac{2}{m}}\beta_{x^i} + 2\beta C^{\frac{2}{m}-1}\beta_0 C_{x^i} - 2mC^{\frac{1}{m}}\beta - \\ & C^{\frac{1}{m}}\beta_0 C_{x^i} + 3\beta^2 C^{\frac{1}{m}-1}\beta_0 C_{x^i} - 3mC^{\frac{1}{m}-1}\beta_0\beta_{x^i}\beta^2 - C^{\frac{4}{m}-1}\sigma_0 C_{x^i} + mC^{\frac{3}{m}}\sigma_0\beta_{x^i} + \\ & \left. 2\beta^2 C^{\frac{2}{m}-1}\sigma_0 C_{x^i} + 2m\beta^2 C^{\frac{1}{m}}\sigma_0\beta_{x^i} - \beta^2 C^{\frac{1}{m}-1}\sigma_0 C_{x^i} - m\beta^2\sigma_0\beta_{x^i} \right]. \quad (45) \end{aligned}$$

Which by multiplying it with y^i , we get

$$\begin{aligned} P_0 = & \frac{1}{2\left(\frac{1}{C^m}-\beta\right)^2} \left[C^{\frac{4}{m}-1}(2\theta C_0 + 2C_0\sigma_0) + C^{\frac{3}{m}-1} \left(2\theta\beta C_0 - \left(4 + \frac{3}{m} \right) m\theta\beta C_0 - 6C_0\sigma_0\beta + \right. \right. \\ & \left. 4\beta_0 C_0 \right) + C^{\frac{2}{m}-1}(4m\theta C_0\beta^2 - 2C_0\beta_0) + C^{\frac{1}{m}-1}(4\beta_0 C_0\beta^2 + 2\beta^3\sigma_0 C_0) + C^{\frac{3}{m}}(-3m\theta_0\beta + \\ & 2m\beta_{00} - 2m\beta\sigma_{00} + m\sigma_0\beta_0 + m\theta\beta_0) + C^{\frac{2}{m}}(-2m\beta^2\sigma_{00} - 4m\beta\beta_{00} + 2m\theta_0\beta^2 - 4\beta_0\beta - \\ & 6m\beta\beta_0\sigma_0 - 2m\beta_0^2 - 2m\theta\beta_0) + C^{\frac{1}{m}}(-m\beta^2\beta_{00} - 4m\beta\beta_0^2 + 2m\beta\sigma_0\beta_0 + 6m\beta^2\sigma_0\beta_0 + \\ & m\beta^2\beta_{00} + 2m\beta^3\sigma_{00} + 2\beta\theta\beta_0 + 2m\beta_0^2 - 2m\beta\beta_0 C_0) + m\theta_0 C^{\frac{4}{m}} + (3m\beta^3 + 6m\beta^2\beta_0^2 + \\ & \left. 3m\beta^3\beta_{00} + 6m\beta^2\beta_0^2 - 2m\beta^2\beta_0\sigma_0 - m\beta^3\sigma_{00} - m\beta^2\beta_0\sigma_0) \right]. \quad (46) \end{aligned}$$

Suppose \bar{F} is of constant flag curvature, then

$$K(x, y)\bar{F}^2 + \bar{P}_0 - \bar{P}^2 = 0, \quad (47)$$

Putting the values of K , \bar{F} , \bar{P}_0 and \bar{P}^2 , we have

$$\begin{aligned}
 K(x, y) & \left\{ \left(\frac{F^2}{F-\beta} \right) + \beta \right\} e^{\sigma(x)} \left\{ 2 + \frac{1}{4 \left(\frac{1}{C_0^m - \beta} \right)^2} \left[C_0^{4-m} (4\theta C_0 + 4C_0\sigma_0) + C_0^{3-m} \left(4\theta\beta C_0 - 2 \left(4 + \right. \right. \right. \right. \\
 & \left. \left. \left. \frac{3}{m} \right) m\theta\beta C_0 - 6C_0\sigma_0\beta + 4\beta_0 C_0 \right) + C_0^{2-m} (8m\theta C_0\beta^2 - 4C_0\beta_0) + C_0^{1-m} (8\beta_0 C_0\beta^2 + \right. \\
 & \left. 4\beta^3\sigma_0 C_0) + C_0^3 (-3m\theta_0\beta + 2m\beta_{00} - 2m\beta\sigma_{00} + m\sigma_0\beta_0 + m\theta\beta_0 + 2m^2\beta^2\theta\beta_0 - \right. \\
 & \left. 16m^2\beta^2\beta_0\sigma_0 + 2\beta^2 m^2_0\sigma^2_0 + 2m^2\beta^2\theta\sigma_0 - 8m^2\beta\beta_0^2) + C_0^2 (-2m\beta^2\sigma_{00} - 4m\beta\beta_{00} + \right. \\
 & \left. 2m\theta_0\beta^2 - 4\beta_0\beta - 6m\beta\beta_0\sigma_0 - 2m\beta_0^2 - 2m\theta\beta_0 + 4m^2\beta^4\sigma^2_0 - 8m^2\beta^2\beta_0^2 + \right. \\
 & \left. 12m^2\beta^3\theta\beta_0 + 8m^2\beta^3\theta\sigma_0 - 4m^2\beta^3\theta\sigma_0 + 12m^2\beta^3\sigma_0\beta_0) + C_0^1 (-m\beta^2\beta_{00} - 4m\beta\beta_0^2 + \right. \\
 & \left. 2m\beta\sigma_0\beta_0 + 6m\beta^2\sigma_0\beta_0 + 3m\beta^2\beta_{00} + 2m\beta^3\sigma_{00} + 2\beta\theta\beta_0 + 2m\beta_0^2 - 2m\beta\beta_0 C_0 + \right. \\
 & \left. 12m^2\beta^3\beta_0^2 - 4m^2\beta^4\sigma_0^2 - 4m^2\beta^3\sigma_0\beta_0 + 12m^2\beta^4\sigma_0\beta_0) + (m\theta_0 + 4m^2\beta_0^2 + \right. \\
 & \left. 2m^2\theta^2\beta^2 - 4m^2\theta\beta\beta_0 - 4m^2\beta^2\sigma_0 - 4m^2\theta\beta\beta_0 - 4m^2\beta^2\theta\sigma_0 - 4m^2\beta\beta_0\sigma_0) C_0^4 + \right. \\
 & \left. C_0^6 (m^2\theta^2 + 2\theta m^2\sigma_0 + m^2\sigma_0^2) + C_0^5 (4m^2\theta\beta_0 - m^2\theta^2\beta + 2m^2\beta_0\sigma_0 - 4\sigma_0\theta\beta m^2) + \right. \\
 & \left. (9m^2\beta^4\beta_0^2 + m^2\beta^4\sigma_0^2 - 6m^2\beta^4\sigma_0\beta_0 + 6m\beta^2\beta_0^2 + 3m\beta^3\beta_{00} + 6m\beta^2\beta_0^2 - \right. \\
 & \left. 2m\beta^2\beta_0\sigma_0 - m\beta^3\sigma_{00} - m\beta^2\beta_0\sigma_0) \right\}. \tag{48}
 \end{aligned}$$

Lemma 4.1: [12] Let $\mathcal{F} = C_0^{\frac{1}{m}}$ ($m > 2$), be an m^{th} -root Finsler metric on an open subset $\mathbb{V} \subset \mathbb{R}^n$. Suppose that the equation $PC_0^{\frac{5}{m-r}} + QC_0^{\frac{4}{m-r}} + RC_0^{\frac{3}{m-r}} + SC_0^{\frac{2}{m-r}} + TC_0^{\frac{1}{m-r}} + U = 0$ where $r = 0, 1, 2, 3, 4, 5, \dots$ and P, Q, R, S, T and U are homogeneous polynomials in y . Then $P = Q = R = S = T = U = 0$.

Using Lemma 4.1, we have $K = 0$ and the following conditions

$$\theta C_0 + 4C_0\sigma_0 = 0, \tag{49}$$

$$4\theta\beta C_0 - 2 \left(4 + \frac{3}{m} \right) m\theta\beta C_0 - 6C_0\sigma_0\beta + 4\beta_0 C_0 = 0, \tag{50}$$

$$(8m\theta C_0\beta^2 - 4C_0\beta_0) = 0, \tag{51}$$

$$(8\beta_0 C_0\beta^2 + 4\beta^3\sigma_0 C_0) = 0, \tag{52}$$

$$2m\beta_{00} - 3m\theta_0\beta - 2m\beta\sigma_{00} + m\sigma_0\beta_0 + m\theta\beta_0 + 2m^2\beta^2\theta\beta_0 - 16m^2\beta^2\beta_0\sigma_0 + 2\beta^2 m^2_0\sigma^2_0 + 2m^2\beta^2\theta\sigma_0 - 8m^2\beta\beta_0^2 = 0, \tag{53}$$

$$(2m\theta_0\beta^2 - 2m\beta^2\sigma_{00} - 4m\beta\beta_{00} - 4\beta_0\beta - 6m\beta\beta_0\sigma_0 - 2m\beta_0^2 - 2m\theta\beta_0 + 4m^2\beta^4\sigma^2_0 - 8m^2\beta^2\beta_0^2 + 12m^2\beta^3\theta\beta_0 + 8m^2\beta^3\theta\sigma_0 - 4m^2\beta^3\theta\sigma_0 + 12m^2\beta^3\sigma_0\beta_0) = 0, \tag{54}$$

$$(m\theta_0 + 4m^2\beta_0^2 + 2m^2\theta^2\beta^2 - 4m^2\theta\beta\beta_0 - 4m^2\beta^2\sigma_0 - 4m^2\theta\beta\beta_0 - 4m^2\beta^2\theta\sigma_0 - 4m^2\beta\beta_0\sigma_0) = 0, \tag{55}$$

$$(m^2\theta^2 + 2\theta m^2\sigma_0 + m^2\sigma_0^2) = 0, \tag{56}$$

$$(4m^2\theta\beta_0 - m^2\theta^2\beta + 2m^2\beta_0\sigma_0 - 4\sigma_0\theta\beta m^2) = 0, \tag{57}$$

$$9m^2\beta^4\beta_0^2 + m^2\beta^4\sigma_0^2 - 6m^2\beta^4\sigma_0\beta_0 + 6m\beta^2\beta_0^2 + 3m\beta^3\beta_{00} + 6m\beta^2\beta_0^2 - 2m\beta^2\beta_0\sigma_0 - m\beta^3\sigma_{00} - m\beta^2\beta_0\sigma_0 = 0. \tag{58}$$

Thus, we conclude that

Theorem: Let $\bar{\mathcal{F}} = \bar{\mathcal{F}}(x, y)$ be the conformal Matsumoto-Randers change of an m^{th} -root Finsler metric $\mathcal{F} = C^{\frac{1}{m}}$ ($m > 4$ and C is irreducible). Suppose $\bar{\mathcal{F}}$ is projectively flat with constant flag curvature. Then $\bar{\mathcal{F}}$ necessarily has vanishing flag curvature $K = 0$, and satisfies the conditions (49)–(58).

Applications

In this section we have given some fruitful applications and future scope of our research paper. Dually flat Finsler metrics, such as those investigated in this paper, play a central role in information geometry, a field closely connected to statistical inference, machine learning, and signal processing. The characterization of projectively and dually flat structures enhances our understanding of geodesic flows, which is fundamental in optimizing information-theoretic learning algorithms.

The study of m^{th} -root Finsler metrics, particularly under conformal Matsumoto-Randers transformations, has significant implications for modeling spacetime geometries in non-Riemannian generalizations of Einstein's relativity. The results on projective flatness and flag curvature contribute to a deeper understanding of spacetime anisotropies, with potential applications in cosmology and unified field theories.

Moreover, the paper addresses the Finslerian analogue of Hilbert's Fourth Problem by characterizing metrics whose geodesics are straight lines (projectively flat). These findings have practical applications in geometric modeling and computational geometry.

The explicit conditions and theorems established in this work provide a solid foundation for

- Constructing new classes of Finsler spaces,
- Investigating additional conformal and projective transformations, and
- Applying these geometrical models in modern mathematical physics, particularly in anisotropic diffusion processes and energy-minimizing path analyses.

References

- [1]. S. I. Amari and H. Nagaoka, "Methods of Information Geometry", AMS Translations Mathematical Monographs, Oxford University Press, 2000.
- [2]. D. Bao and Z. Shen, "Finsler metrics of constant positive curvature on the Lie group S^3 ", Journal of London Mathematical society, 66 (2) (2002), 453-467.
- [3]. R. Bryant, "Projectively flat Finsler 2-spheres of constant curvature", Selecta Mathematica 3(2) (1997), 161-203.
- [4]. S. S. Chern and Z. Shen, "Riemannian-Finsler geometry", Singapore, World Scientific, (2005).
- [5]. G. Hamel, "Über die Geometrien in denen die Geraden die Kurzesten sind"., Math. Ann., 57 (1903), 231-264.
- [6]. M. Kumar and C. K. Mishra, "On conformal Kropina transformation of m^{th} -root metrics", Journal of the Indian Math. Soc., 88(1-2) (2021), 97-104.
- [7]. M. Kumar, "Locally dually flatness and locally projectively flatness of Matsumoto change with m^{th} -root Finsler metrics", Journal of Finsler Geometry and its Applications, 2(1) (2021), 31-39.

- [8]. Z. Shen and Q. Xia, “A class of Randers metrics of scalar flag curvature”, *Int. J. Math.*, 24(7) (2013), 1350055.
- [9]. Z. Shen, “Differential geometry of spray and Finsler spaces”, Springer, 2013.
- [10]. H. Shimada, “On Finsler space with the metric $L = \sqrt[m]{a_{i_1 i_2 i_3 i_4 \dots i_m} y^{i_1} y^{i_2} y^{i_3} y^{i_4} \dots y^{i_m}}$ Tensor N.S., 33(3) (1979), 365-372.
- [11]. A. Tayebi, M. Shahbazi Nia, and E. Peyghan, “On generalized m-th root Finsler metrics”, *Linear Algebra and its Applications*, 437 (2012), 675-683.
- [12]. A. Tayebi and B. Najafi, “On m-th root Finsler metrics”, *Journal of Geometry and Physics*, 8(1) (2015), 14-20.
- [13]. B. Tiwari, M. Kumar and A. Tayebi, “On generalized Kropina change of generalized mth-root Finsler metrics”, *Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci.*, 91(3) (2021), 443-450.
- [14]. P. K. Dwivedi, C. K. Mishra and S. Kumar, “On projectively flat Finsler space with n-power (α, β) -metric”, *Journal of Finsler Geometry and its Applications*, Vol. 5(2) (2024), 14-24.
- [15]. U. Shanker, A. Singh, P. K. Dwivedi, C. K. Mishra, R. K. Pandey and S. Kumar, “Special Finsler spaces with an exponential (α, β) -metric”, Accepted in *Journal of Calcutta Mathematical Society*.
- [16] M. K. Gupta and A. Tayebi, “On conformal-Matsumoto change of mth-root Finsler metrics”, *Facta University (NIS) Ser. Math. Inform.* Val. 38 (5) (2023), 895-903.