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DYNAMIC ROUTH'S STABILITY FOR NONLINEAR DYNAMIC SYSTEMS USING DPMA

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Abstract: *In this paper, an effective method is proposed called the "Dynamic Routh's stability Criterion (DRSC)", which is developed using the Dynamic Pole Motion (DPM) approach (DPMA). This innovative technique extends the classical Routh's stability criterion—traditionally limited to linear time-invariant (LTI) systems—to encompass a broader range of systems, including linear time-varying and nonlinear dynamic systems. By incorporating the behavior of pole trajectories over time, the proposed method offers a more comprehensive framework for analyzing system stability in more complex and realistic scenarios. DPM approach describes the notion of dynamic poles (DP) in a three dimensional 'g-plane', which is an extension to the two dimensional 's-plane'. The s-domain approach is based upon Laplace transform and has been used for linear time-invariant (LTI) system only. This novel g-plane framework is versatile enough to be applied to both linear and nonlinear dynamic systems. For the stability of dynamic systems, the DP must present in the left-hand side (LHS) of the g-plane. For nonlinear systems, the locations of DP are the function of system states; and these systems states are the function of initial conditions with amplitude and frequency (AnF) of input signals. Therefore, in nonlinear systems, stability is influenced not only by the initial conditions but also by the AnF of the input signals. For example, for given amplitude of the input signal, system may be unstable at low frequency; however, it may become stable at high frequency or vice-versa. These stability conditions are illustrated by several examples.*

Keywords: *Dynamic Routh's stability (DRS), nonlinear dynamic systems, dynamic poles (DP), Dynamic Routh's array, Dynamic pole motion (DPM) approach.*

1. Introduction

Nonlinear behavior is a natural and common trait in many dynamic systems encountered in the real world. This characteristic often arises from the physical nature of system components or the interactions involved in their motion—such as the forces experienced in rotating mechanisms. These effects introduce complexities that linear models cannot fully capture, making nonlinear analysis essential for understanding how such systems behave under various conditions.

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In contrast, linear dynamic systems are generally simpler to study. They usually have a single equilibrium point, located at the origin of the phase plane. The stability of this point can be easily evaluated using well-established linear techniques. Depending on the system's parameters, this equilibrium point can either be stable—drawing system responses toward it—or unstable, causing small disturbances to grow over time.

Nonlinear systems, however, present a much richer and more complex picture. They can have multiple equilibrium points located at different positions in the phase space. Each of these points may behave differently: some can act as attractors, where the system naturally settles into a steady state, while others can cause the system to drift away, indicating instability. Because of this diversity in behavior, nonlinear systems require more sophisticated analytical approaches, as conventional linear tools are often insufficient to capture the full range of their dynamics.

There are various approaches to deal with non-linear systems stability analysis [1, 8], however, in this section we briefly describe only some of these approaches.

(i) Method of Linearization in the Neighbourhood of the Operating Point [1, 8, 10]

To understand the behavior of nonlinear systems around specific conditions, a common strategy involves approximating the nonlinear system with a linear one near a chosen operating point. This method is practical because the stability of nonlinear systems is typically local, meaning it depends on how the system behaves in the immediate surroundings of an equilibrium.

One of the most widely used techniques for this is the Jacobian linearization. In this approach, a Jacobian matrix is computed at the operating point—often denoted (x_0, y_0). The eigenvalues of this matrix indicate the local stability: if all eigenvalues have -ve real parts, the point is locally stable. If even one has a +ve real part, the equilibrium is unstable.

Unlike linear systems, which usually have just one equilibrium point, nonlinear systems can have multiple equilibrium points—some stable, some not. To visualize these, phase-plane analysis is used. In this method, if the system's trajectories in the phase plane move toward an equilibrium point, that point is considered stable; if they move away, it indicates instability.

(ii) **Lyapunov's Stability Method [8, 9, 10]:**

Lyapunov's direct method is a powerful tool for evaluating the stability of nonlinear

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systems without having to solve the system equations directly. For a system expressed as $\dot{x} = f(x)$, a Lyapunov function V(x) is proposed. This function is analogous to the concept of energy in physics: it must be positive definite, meaning it is always positive except at the equilibrium, where it is zero.
To confirm that the system is stable, the time derivative of V(x), denoted $\dot{V}(x)$, must be negative definite—this ensures that the system's "energy" is decreasing over time. If $\dot{V}(x)$ is only negative semi-definite, the system may be oscillatory but still bounded. This method allows one to infer stability based on the behavior of the candidate function and its rate of change.

(iii) Popov's Stability Criterion [10]:

The Popov criterion is a graphical and analytical method for determining the system robustness of non-linear control systems, particularly those with feedback involving non-linear components. It draws parallels to the Nyquist criterion used for linear systems.

Imagine a non-linear system whose transfer function comprises both a linear component G(jω) and a non-linear element characterized by a slope k. According to the Popov criterion, the system is globally asymptotically stable if there exists a real number q (which can be any real value) such that the following condition holds true for all frequencies ω:

$$\mathrm{Re}\,[(1 + j\omega q)\,G(j\omega)] + 1/k > 0$$

This condition can be interpreted graphically as well. In the frequency domain, the plot of G(jω) must lie entirely to the right of a line called the Popov line, which has a slope of -1/q and crosses the real axis at -1/k. If this graphical condition is met across all frequencies, the system is considered stable.

(iv) LaSalle's Invariance Principle [6-10]:

Barbashin-Krasovskii-LaSalle principle is a significant extension of Lyapunov's method. It is particularly useful for assessing the asymptotic behavior of autonomous nonlinear systems when the derivative of the Lyapunov function is not strictly negative.

The principle states that even if the derivative of the Lyapunov function is only negative semi-definite, the trajectories of the system will eventually converge to the largest invariant

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set where $V'(x) = 0$. This makes the LaSalle principle a powerful tool for proving asymptotic stability in cases where Lyapunov's strict condition is not satisfied. It has been applied widely in stability analysis, especially when dealing with complex or loosely damped systems.

The structure of this paper is as follows: Section 2 introduces the concept of DPM in the g-plane, specifically developed for analyzing dynamic systems. In Section 3, we present a detailed explanation of the DRSC as it applies to nonlinear dynamic systems. To demonstrate the effectiveness and robustness of this extended stability analysis technique, Section 4 provides a series of illustrative examples. Finally, Section 5 concludes the paper with key findings and insights derived from the proposed approach.

2. Dynamic Pole Motion Approach:

2.1 Classification of *Dynamic Systems*:

Let us consider a general class of dynamic system shown in Figure 1 and mathematically expressed as-

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}\big(\boldsymbol{x}(t), \mathbf{u}(t)\big), \boldsymbol{x}(0) \\ \boldsymbol{y}(t) = \boldsymbol{h}(\boldsymbol{x}(t), \boldsymbol{u}(t)) \end{cases} \tag{1}$$

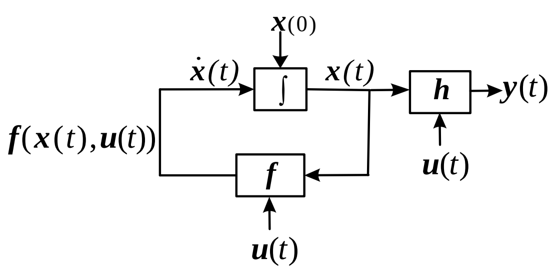

Figure 1. A general dynamic system (Eqn.1)

Now, the nonlinear dynamic system can be represented in the state-space form as;

$$\begin{cases} \dot{\boldsymbol{x}}(\boldsymbol{t}) = \boldsymbol{A}(\boldsymbol{x}, t)\boldsymbol{x}(t) + \boldsymbol{B}(t)\boldsymbol{u}(t),\ \ \boldsymbol{x}(\boldsymbol{0}) \\ \boldsymbol{y}(\boldsymbol{t}) = \boldsymbol{C}(t)\boldsymbol{x}(t) + \boldsymbol{D}(t)\boldsymbol{u}(t) \end{cases} \tag{2}$$

where,

$\boldsymbol{x}(t) \in \mathrm{R}^{\mathrm{n}}$: vector of system states, $\dot{\boldsymbol{x}}(t) \in \mathrm{R}^{\mathrm{n}}$: time derivative of $\boldsymbol{x}(t)$

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$\boldsymbol{x}(0) \in \mathrm{R}^{\mathrm{n}}$: vector of initial conditions, $\boldsymbol{u}(t) \in \mathrm{R}^{m}$: vector of system inputs,
$\boldsymbol{y}(t) \in \mathrm{R}^{p}$: vector of system output, $\boldsymbol{A}(\boldsymbol{x}, t) \in \mathrm{R}^{n\times n}$: state matrix,
$\boldsymbol{B}(t) \in \mathrm{R}^{n\times m}$: input matrix, $\boldsymbol{C}(t) \in \mathrm{R}^{p\times n}$: output matrix, and
$\boldsymbol{D}(t) \in \mathrm{R}^{p\times m}$: feed-forward matrix

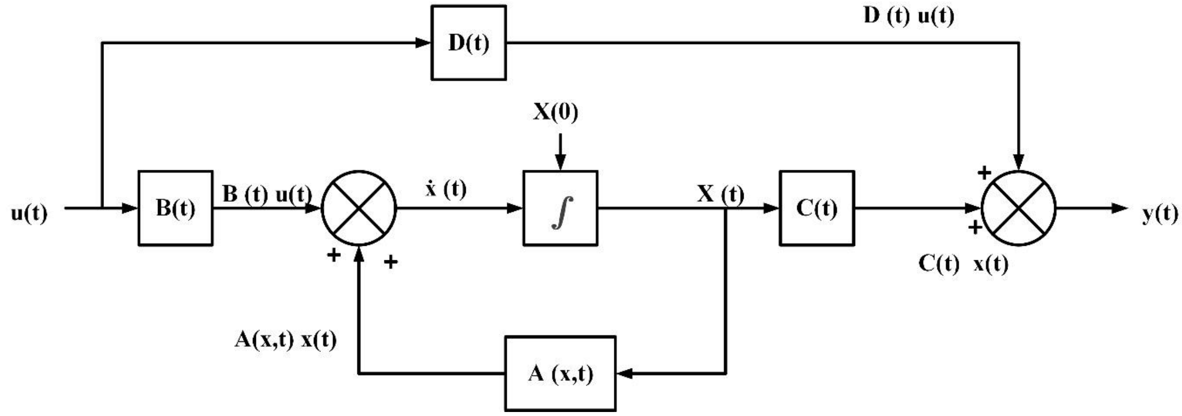

Figure 2. Model of a dynamic nonlinear system (Eqn. 2)

Dynamic systems, in general, can be classified into the following categories:

(i) Linear systems

(i-a) Linear time-invariant (LTI) systems: system matrix $\boldsymbol{A}$ has constant elements;

(i-b) Linear time-varying (LTV) systems: system matrix $\boldsymbol{A}(\boldsymbol{t})$ has some time- varying elements;

(ii) Nonlinear systems: system matrix $\boldsymbol{A}(\boldsymbol{x}, t)$ has some elements which are the function, explicitly or implicitly, of the state x and time t.

A linear time-invariant system (LTI) represented as

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \quad (3)$$

For an autonomous system, $\boldsymbol{u}(t) = \boldsymbol{0}$ and this is represented as

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = \boldsymbol{A}\boldsymbol{x}(t) \quad (4)$$

In LTI, the parameters of matrix $\boldsymbol{A}$ are constant (independent of x and t) i.e. matrix $\boldsymbol{A}$ for 2nd order system that can be represented as -

$$\boldsymbol{A} = [a_{ij}]_{2\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 7 & 5 \end{bmatrix}$$

.

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For *linear time-varying system*, however the state matrix $\boldsymbol{A}$ will have one or more time-varying parameters i.e. matrix $\boldsymbol{A}(\boldsymbol{t})$ that can be represented as -

$$\boldsymbol{A}(\boldsymbol{t}) = \left[a_{ij}(t)\right]_{2\times 2} = \begin{bmatrix} 2t & 4 \\ 7 & 5 \end{bmatrix}$$

For *nonlinear time-invariant system, the* parameters of the state matrix $\boldsymbol{A}$ are explicitly function of $\boldsymbol{x}$; i.e. matrix $\boldsymbol{A}(\boldsymbol{x})$ that can be represented as -

$$\boldsymbol{A}(\boldsymbol{x}) = \left[a_{ij}(x)\right]_{2\times 2} = \begin{bmatrix} 2 & 4 \\ (7-x) & 5 \end{bmatrix}$$

For *nonlinear time varying systems, the p*arameters of matrix $\boldsymbol{A}$ are explicitly function of $\boldsymbol{x}$ and $\boldsymbol{t}$; i.e. matrix $\boldsymbol{A}(\boldsymbol{x},\boldsymbol{t})$ that can be represented as -

$$\boldsymbol{A}(\boldsymbol{x},\boldsymbol{t}) = \left[a_{ij}(x,t)\right]_{2\times 2} = \begin{bmatrix} 2 & 4 \\ 7-x & 5t \end{bmatrix}$$

2.2 *Introduction to the g-plane*:

In this section, a novel analytical framework known as the g-plane is introduced, specifically developed to facilitate the study of nonlinear dynamic systems. Unlike traditional methods that rely solely on the s-plane for analyzing system behavior, the g-plane offers a more flexible and comprehensive view. Within this framework, the system's characteristic equation is formulated using a specialized differential operator denoted as g. The solutions to this equation—referred to as dynamic poles—represent key indicators of system behavior and are mapped within the g-plane. These dynamic poles vary over time and system states, providing valuable insights into how nonlinear systems evolve and maintain or lose stability.

Conventionally, for LTI systems we use the Laplace differential operator 's' defined as, $s \triangleq d/dt$, and the roots of the Char. Eqn. are expressed in a 2-dimensonal s-plane, $s = \sigma + j\omega$. For a general class of systems, both linear and nonlinear, we introduce a new differential operator,

$$g \triangleq d/dt \quad (5)$$

and, define the g-plane as

$$g(t) = \sigma(t) + j\omega(t) \quad (6)$$

where, σ *and* $j\omega$ are, explicitly or implicitly, function of time t. For example, it may be an implicit function of time through the state variable $\boldsymbol{x}(t)$. The three-dimensional g-plane is shown in Fig.3.

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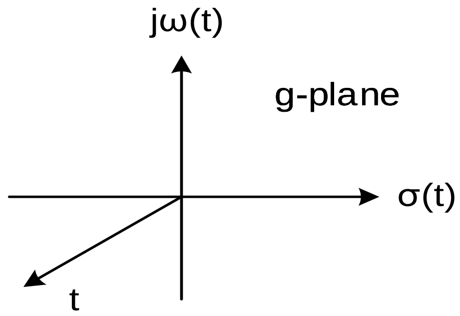

Figure 3. The three dimensional g-plane; $g(t) = \sigma(t) + j\omega(t)$

In the g-plane defined in Eqn. (6) and shown in Fig.3, $\sigma(t)$ and $j\omega(t)$ are the real and imaginary axis with an explicit function of time t.

Note that for the conventional Laplace operator 's', which is being used only for LTI systems, the s-plane is only a two-dimensional plane with σ and $j\omega$ axes (without time axis). Whereas, the 'g' operator and g-plane with $\sigma(t)$, $j\omega(t)$ and t axes, as explained in this paper, can be used for both linear & nonlinear systems. *Hence, the s-plane can be considered a subset within the broader framework of the g-plane.*

Using the differential operator 'g', the dynamic Char. Eqn. for the nonlinear system given in Eqn. (2) is defined as

$$|gI - \boldsymbol{A}(x,t)| = 0 \quad (7)$$

The *dynamic Char. Eqn.;* Eqn. (7) is also expressed as

$$a_n(x,t)g^n + a_{n-1}(x,t)g^{n-1} + \cdots \dots \dots .. + a_0(x,t) = 0 \quad (8)$$

The roots of this nth-order Char. Eqn. are

$$\lambda_i(x,t); i = 1, \dots .. n \quad (9)$$

In nonlinear systems, the characteristic roots—denoted as λ_i(x, t)—depend not just on time t, but also on the system's state x(t). Because of this dependency, these roots are referred to as dynamic poles (D.P.). Now, since the state x(t) itself is influenced by both the initial condition x(0) and the i/p signal u(t), the

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location of these DP isn't fixed. The input signal u(t) typically varies in terms of amplitude and frequency, which means that in nonlinear systems, the position of dynamic poles is directly affected by the initial state and the characteristics of the input.

This is a key difference from linear systems, where the pole locations—and thus system stability—are independent of the input or starting condition. In nonlinear systems, however, the dynamic behavior and stability can change based on how the system is started and what kind of input it receives.

The way these poles shift in g-plane over time, as the state x(t) evolves, is described by what's known as the dynamic root-locus (DRL). This concept provides valuable insight into system behavior. If the DP remain on the LHS of the g-plane, the system remains stable. If they drift into the right side, the system becomes unstable. If the poles are positioned along the img-axis jω(t), the system is marginally stable.

In the upcoming section, we'll explore how a general method DRS method can be used to assess the stability of both linear and nonlinear systems based on the motion of these DP.

3. Dynamic Routh's Stability Method for Dynamic Systems:

Traditionally, the Routh's stability criterion was developed using the differential operator "s" from the Laplace domain, making it applicable only to linear time-invariant (LTI) systems. However, this approach falls short when dealing with more complex systems that exhibit nonlinear or time-varying behavior. To address this, the current section introduces an alternative framework based on the g-plane, as discussed earlier in Section 2.1. This extended method DRSC adapts the classical Routh technique to work within the g-plane environment. As a result, it enables stability analysis not only for LTI systems but also for linear time-varying and nonlinear dynamic systems, offering a more generalized and robust approach.

***3.1 Formation of DRA*:**

The stability requirements for the nonlinear dynamic system given in Eqn. (2) must now be ascertained, we formulate the *DRA* for the dynamic Char. Eqn. given in Eqn. (8) as follows:

$$g^n \;\Big|\quad a_n \qquad a_{n-2} \qquad a_{n-4} \qquad a_{n-6} \qquad \cdots$$

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$$\begin{array}{c|ccccc} g^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\ g^{n-2} & b_1 & b_2 & b_3 & \cdots & \cdots \\ g^{n-3} & c_1 & c_2 & c_3 & \cdots & \cdots \\ g^{n-4} & \vdots & \vdots & \vdots & & \\ \vdots & \vdots & \vdots & \vdots & & \\ g^1 & \vdots & \vdots & \vdots & & \\ : g^0 & \vdots & \vdots & \vdots & & \end{array} \quad (10)$$

where, $a_i = a_i(\boldsymbol{x}, t)$

The first two rows of the Dynamic Routh Array (DRA), as defined in Equation (10), are constructed using the even and odd coefficients of the characteristic polynomial, respectively. The subsequent rows are systematically derived from these initial two rows using a recursive process.

In this DRA, b_j are the elements in the third row, and $c_j$are the elements in the fourth row etc. These row elements are defined as:

$$\begin{cases} b_1 = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} = -\frac{(a_n a_{n-3} - a_{n-1} a_{n-2})}{a_{n-1}} \\ b_2 = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix} = -\frac{(a_n a_{n-5} - a_{n-1} a_{n-4})}{a_{n-1}} \\ b_3 = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-6} \\ a_{n-1} & a_{n-7} \end{vmatrix} = -\frac{(a_n a_{n-7} - a_{n-1} a_{n-6})}{a_{n-1}} \end{cases} \quad (11)$$

$$\begin{cases} c_1 = \frac{-1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix} = -\frac{(a_{n-1} b_2 - b_1 a_{n-3})}{b_1} \\ c_2 = \frac{-1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix} = -\frac{(a_{n-1} b_3 - b_1 a_{n-5})}{b_1} \end{cases} \quad (12)$$

The third row element onwards of DRA can be computed similarly with Routh's array element as shown in Eqn. 11 &12.

The non-linear dynamic system stability can be determined by DRA as described below.

If a zero appears in the first column of the DRA, it indicates that one or more poles are located on the img-axis jω(t) of the g-*plane. Specifically, if the last two elements in that column are zero, it suggests that both poles are positioned at the origin. On the other hand, if the zero occurs somewhere between the first and*

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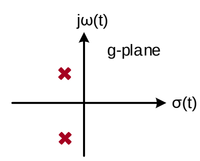
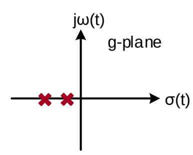
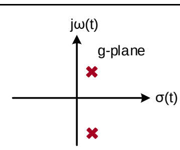
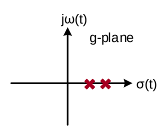
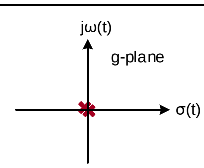
last elements, it implies that the system has a pair of complex conjugate poles lying along the imaginary axis of the g-plane.

According to the DRSC, a nonlinear system is considered stable when all elements in the first column of the DRA, as described in Equation (10), are positive definite.

The various stability conditions for nonlinear systems are given in Table.1.

Table 1: Dynamic Routh's stability conditions

S. No.	Elements of first column of DRA	Location of poles (for second -order system)	Stability of the system
1.	If, all elements are positive definite.	  Then, both poles will lie on left side of g-plane.	Stable
2.	If, two elements sign change	  Then, both poles will lie on the right side of g-plane.	Unstable
3.	If last two elements are zero	 Then, both poles will lie at the origin.	Conditional stable

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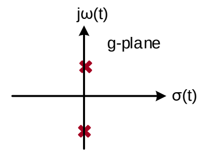
4.	If one element is zero except last element	 Then, poles will be complex conjugate and lie on jω(t) axes.	Conditional stable

The nonlinear systems' stability is dependent upon input vector $\boldsymbol{u}(t)$and initial conditions $\boldsymbol{x}(\mathbf{0})$. The input vector $\mathbf{u}(t)$ depends on frequency (ω) & amplitude of the input signal thus frequency & amplitude of the input affects stability of a non-linear system as shown in Example 1.

4. Some Examples for Stability Analysis for Non-linear Dynamic Systems by using the DRS Method

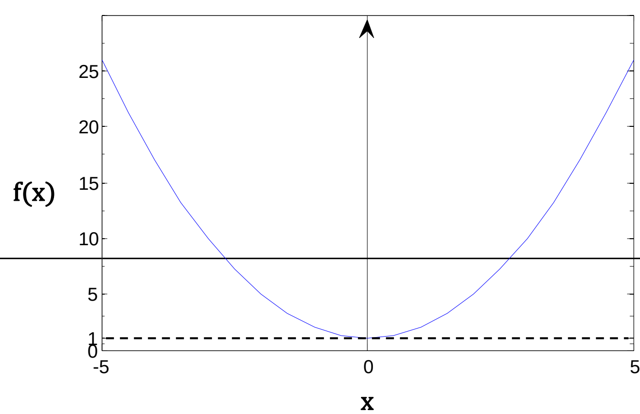
This section presents a stability analysis of several non-linear dynamic systems using DRS Method.

For example, consider a second order non-linear system

$$\ddot{x} + 4\,\dot{x} + f(x)\,x = Asin\omega t \quad , x(0) \qquad (13)$$

Note that in this example, $2\xi\omega_n^2 = 4$, and $\omega_n^2 = f(x), a\ nonlinear\ function$. Now we will consider various types of nonlinearities as shown in Table 2 and determine their stability condition.

Table 2: Some typical types of nonlinear elements in the first column of DRA

S. No.	Nonlinear element f(x) in the first column of DRA	function (f(x)) curve	Type of nonlinear element (f(x))
1.	$f(x) = (1 + x^2)$		Positive definite (for all values of x,

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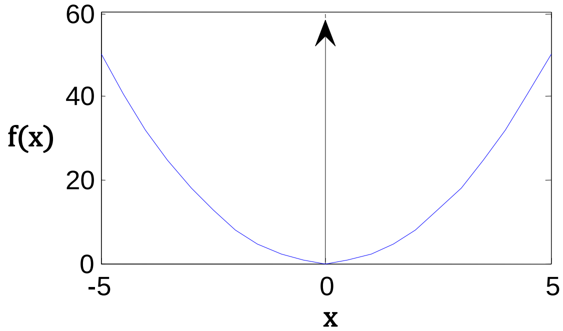
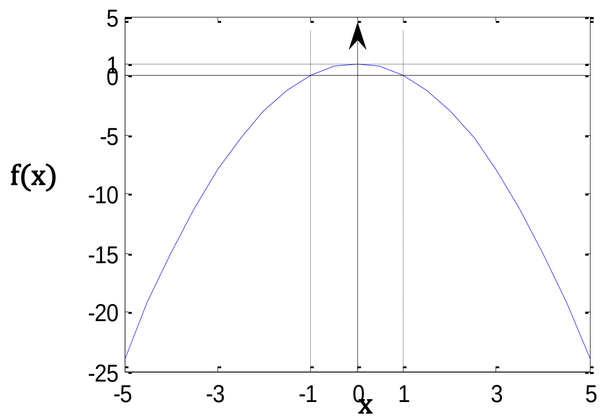
			System is always stable)
2.	$f(x) = x^2$		Positive Indefinite (For x=0, f(x) =0 and System is stable for x > 0)
3.	$f(x) = (1 - x^2)$		Indefinite (for f(x) >0 System is stable)

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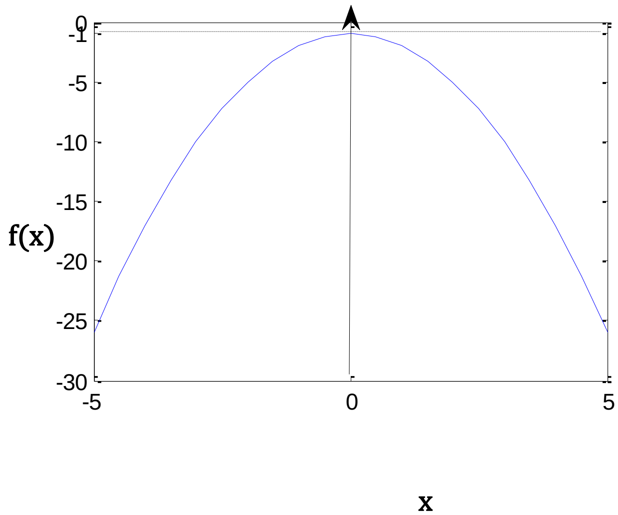
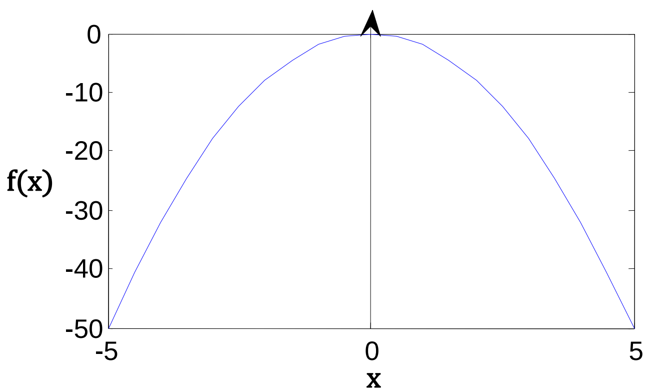
4.	$f(x) = -(1 + x^2)$		Negative definite (for all values of x, System is always unstable)
5.	$f(x) = -x^2$		Negative indefinite (for $x < 0$, System is unstable)

Example 1.

Let us examine a second-order nonlinear system that includes a spring with a nonlinear stiffness characteristic, as illustrated in Figure 4, and is described as,

$$\ddot{x} + 1.1\,\dot{x} + (5 + x)x = Asin\omega t \tag{14}$$

In this example, $u(t) = Asin\omega t$ is the input signal. If input $u(t) = 0$ then the system is called an autonomous nonlinear system as given (Eqn. 15)-

$$\ddot{x} + 1.1\dot{x} + (5 + x)x = 0 \tag{15}$$

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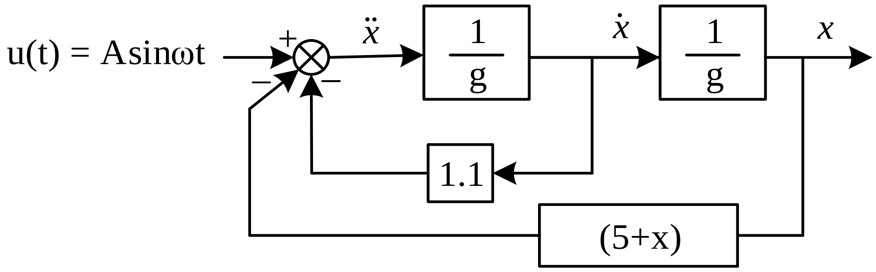

Figure 4. Nonlinear second-order system (Eqn.14)

For the stability of this 2nd order system, all the coefficients in Eqn. (14) must be positive definite. The general 2nd order equation

$a_2\ddot{x} + a_1\dot{x} + \ a_0x = u(t)$ (16)

By comparing Eqn.14 & 16, the coefficients are: a_1 =1.1, a_2 = 1; both are positive definite and

a_0 = (5+x) (a_0 is positive definite only for x > -5)

Figures 5 and 6 demonstrate how the stability of the non-linear system can vary depending on both the frequency and the amplitude of the input signal.

For ω=1 rad/sec, the system remains stable when the input signal amplitude is A=2, as shown in Figure 5(a). However, as illustrated in Figure 5(b), the system becomes unstable when the amplitude increases to A=10. This behavior, consistent with Equation (14), indicates that the dynamic poles (D.P.) have shifted to the RHS of the g-plane, resulting in system instability.

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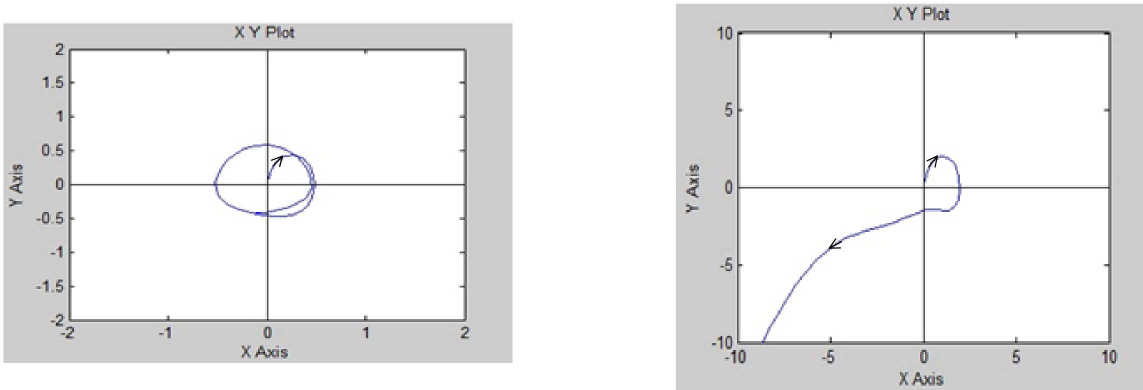
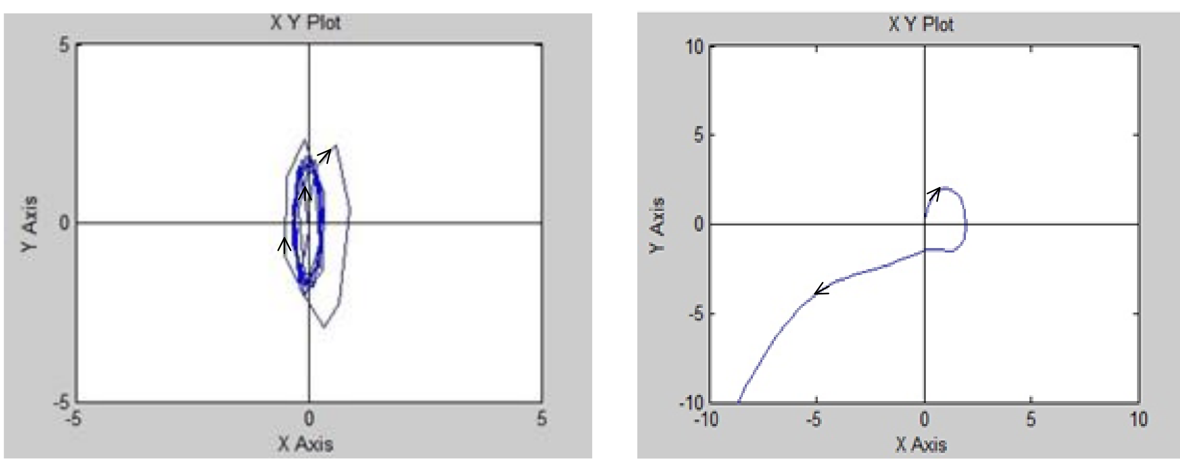


(a) For A=2, and $\omega = 1$rad/sec; stable **(b) For A=10, and $\omega = 1$rad/sec; unstable**

Figure 5. Trajectory for the system in Eqn. (14) ($u(t) = Asin\omega t$)

The stability of the system is changed when there is variation in frequency as shown in the Fig.6(a) & Fig.6(b), the system (Eqn.14) is stable (Fig.6(a)) when input signal frequency ($\omega = 6$ rad.) and unstable (Fig.6(b)) for frequency ($\omega = 1$ rad.), while other parameters are same. So, the stability of the system is varying with change in frequency of input signal. Thus, the system is stable for high frequency and unstable for low frequency.



(a) For A=10, and $\omega = 6$rad/sec; stable **(b) For A=10, and $\omega = 1$rad/sec; unstable**

Figure 6. Trajectory for the system in Eqn.(10) ($u(t) = Asin\omega t$)

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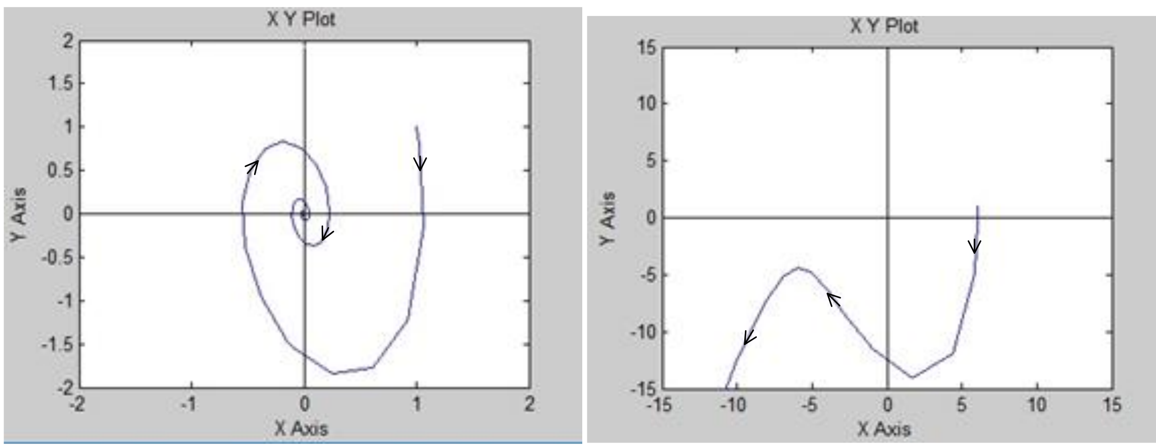
The stability of the system is also changed when there is change in the initial conditions to the system. The Fig.7 shows the change in the stability of the system as varying the initial conditions. The system is stable for initial conditions (1, 1) as shown in Fig.7(a) and unstable for initial conditions (6, 1) in Fig.7(b), while other parameters are same. The stability is guaranteed if, the trajectory converse into $(5 + x)$ region for the system (Eqn.11). if the trajectory converse at equilibrium point then system is unstable if it diverse than the system is unstable.



(a) Stable **(b) Unstable**

Figure 7. Trajectory for the system in Eqn. (11) $(u(t) = 0)$ (a) for initial condition (1, 1) (b) for initial condition (6, 1)

Example 2. A nonlinear system is expressed as -

$$\ddot{x}_1 + 3\,\dot{x}_1 + 2x_1 + x_1^2 = 0 \tag{17}$$

In state space form

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -(2 + x_1)x_1 - 3x_2 \end{cases} \tag{18}$$

By using dynamic Routh's stability method

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(2 + x_1) & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{19}$$

$$\text{Where, } \boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -(2 + x_1) & -3 \end{bmatrix}$$

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The dynamic Char. Eqn. $|gI - \boldsymbol{A}| = \begin{vmatrix} g & -1 \\ (2 + x_1) & g + 3 \end{vmatrix} = 0$

$$g^2 + 3\,g + (2 + x_1) = 0 \tag{20}$$

Dynamic Routh's stability matrix is

$$\begin{array}{c|cc} g^2 & 1 & 2 + x_1 \\ g^1 & 3 & 0 \\ g^0 & 2 + x_1 & 0 \end{array} \tag{21}$$

The first column of DRA should be positive definite (no sign change) to be stable. The first column of DRA (Eqn.21) is positive, but the function element $(2 + x_1)$ is indefinite so the system's stability is not guaranteed.The system's stability is guaranteed when the trajectory of the system cross the boundary of x_1(position axis) and converse at the equilibrium point but, If trajectory doesn't cross the boundary of x_1(position axis) and diverse towards infinity then system is unstable. As shown in Figure 8, the dynamic poles (D.P.) remain on the LHS of the g-plane for $(x_1 > -2)$, indicating that the system is stable. The corresponding dynamic root locus in Figure 8 illustrates how the locations of the D.P. vary with changes in x_1.

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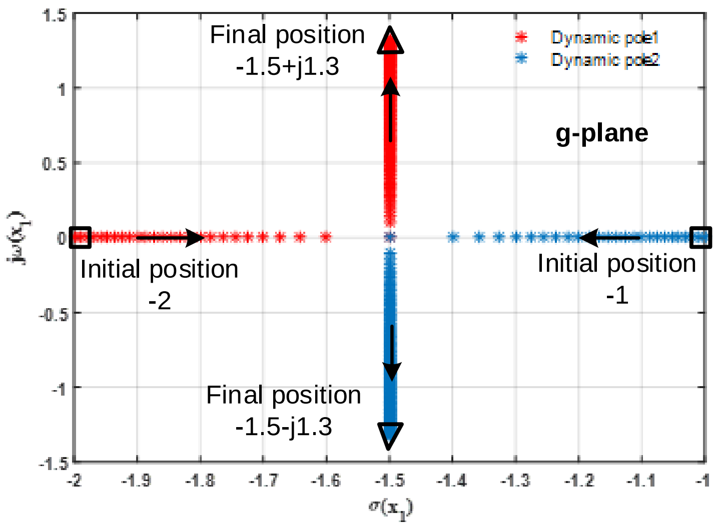
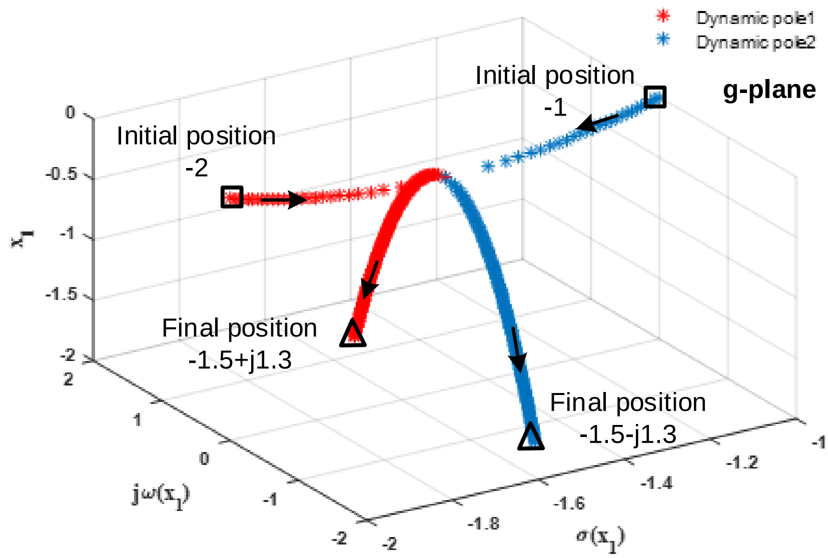


(a)



(b)

Figure 8. Dynamic Root Locus for Eqn.17 showing dynamic pole motion approach for stability (a) 2-D (b) 3-D

Example 3. An equation of nonlinear system with hard spring is given as

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$$\ddot{x}_1 + \dot{x}_1 + (1 + x_1^2)x_1 = 0 \quad (22)$$

by using dynamic-Routh's stability method for the Eqn.22 as follows-

$$\begin{cases} \dot{\boldsymbol{x}}_{\boldsymbol{1}} = x_2 \\ \dot{\boldsymbol{x}}_{\boldsymbol{2}} = -(1 + x_1^2)x_1 - x_2 \end{cases} \quad (23)$$

$$\begin{bmatrix} \dot{\boldsymbol{x}}_{\boldsymbol{1}} \\ \dot{\boldsymbol{x}}_{\boldsymbol{2}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(x_1^2 + 1) & -1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_{\boldsymbol{1}} \\ \boldsymbol{x}_{\boldsymbol{2}} \end{bmatrix} \quad (24)$$

Where,

$$\begin{bmatrix} 0 & 1 \\ -(1 + x_1^2) & -1 \end{bmatrix} = \boldsymbol{A}$$

The dynamic Char. Eqn.; $|gI - \boldsymbol{A}| = \begin{vmatrix} g & -1 \\ (1 + x_1^2) & g + 1 \end{vmatrix} = 0$

$$g^2 + g + (1+x_1^2) = 0 \quad (25)$$

DRA matrix is

$$\begin{array}{c|cc} g^2 & 1 & (1 + x_1^2) \\ g^1 & 1 & 0 \\ g^0 & (1 + x_1^2) & \end{array} \quad (26)$$

The system is considered stable if all elements in the first column of the Dynamic Routh Array (DRA) matrix are positive definite. The function element$(1 + x_1^2)$ is positive definite for all values of x_1 as $(x_1^2 > -1)$. The system's stability is confirmed.

The DP are moving on the left-hand side of g-plane for all values of x_1 and system is stable as shown in Fig.9.

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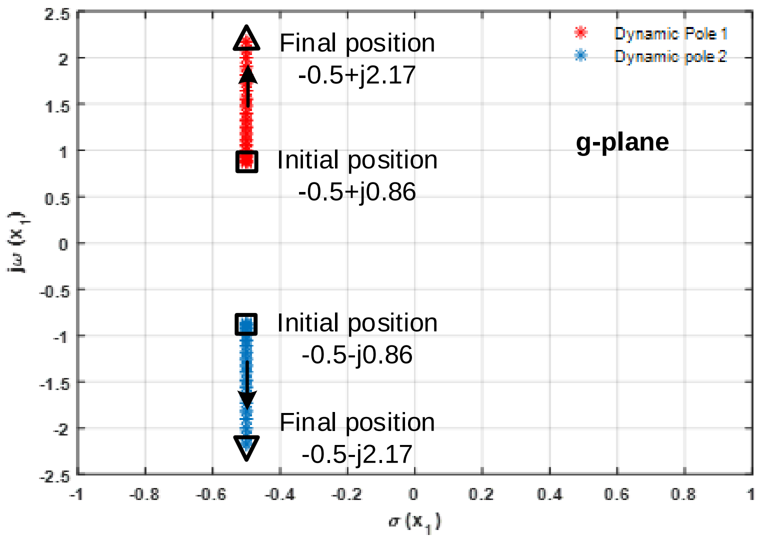
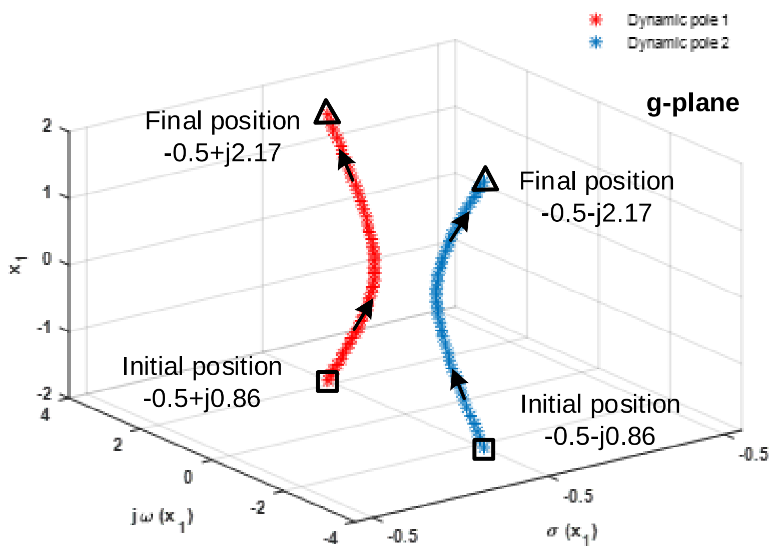


(a)



(b)

Figure 9. Dynamic Root Locus for Eqn.25 showing dynamic pole motion approach for stability (a) 2-D (b) 3-D

Example 4. Let a dynamic Char. Eqn. of a system is given as

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$$g^4 + g^3 + g^2 + g + (x^2 + 3) = 0 \tag{27}$$

The DRA

g^4	1	1	$(x^2 + 3)$
g^3	1	1	
g^2	$b_1 = \epsilon$	$b_2 = (x^2 + 3)$	
g^1	$c_1 = \frac{\epsilon - (x^2 + 3)}{\epsilon}$	$c_1 = 0$	
g^0	$d_1 = (x^2 + 3)$		

(28)

$$b_1 = \frac{1 \times 1 - 1 \times 1}{1} = 0 \rightarrow \epsilon \;, \quad (\epsilon > 0)$$

The elements of the subsequent rows are also determined in terms of ϵ to ascertain the stability conditions for the nonlinear dynamic system stated in Eqn. (2). In this instance, the zero is substituted with a small value ϵ (ϵ>0).

$$b_2 = \frac{1 \times (x^2 + 3) - 1 \times 0}{1} = (x^2 + 3) \tag{29}$$

When evaluating the stability of a non-linear system using the DRA, the first column of the array serves as a key indicator. If a zero appears in this column—as shown in Equation (28)—it suggests that a dominant pole lies on the img-axis of the g-plane. This condition typically leads to continuous oscillations, indicating that the system is marginally stable and on the verge of instability.

Additionally, each sign change in the first column corresponds to a dominant pole located in the RHS of the g-plane. Such poles are associated with system responses that grow exponentially over time, ultimately resulting in instability. As a result, monitoring sign changes in this column offers valuable insight, making it a critical tool for diagnosing and mitigating instability during the design and evaluation of nonlinear systems.

The system (Eqn.27) is stable for the positive definite value of b_1, c_1, d_1 (Eqn. 28)

Example 4. Consider a two-neuron system described by

$$\begin{cases} \dot{x_1} = -x_2 \\ \dot{x_2} = x_1 + (x_1^2 - 1)x_2 \end{cases} \tag{30}$$

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In state space form,

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & (x_1^2 - 1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{31}$$

Where, $\boldsymbol{A} = \begin{bmatrix} 0 & -1 \\ 1 & (x_1^2 - 1) \end{bmatrix}$

The dynamic Char. Eqn.; $|gI - \boldsymbol{A}| = \begin{vmatrix} g & 1 \\ -1 & g - (x_1^2 - 1) \end{vmatrix} = 0$

$$g^2 - (x_1^2 - 1)g + 1 = 0 \tag{32}$$

$$g^2 + (1 - x_1^2)g + 1 = 0 \tag{33}$$

DRA is

$$\begin{array}{c|cc} g^2 & 1 & 1 \\ g^1 & (1 - x_1^2) & 0 \\ g^0 & 1 & 0 \end{array} \tag{34}$$

The DRA (Eqn.34) shows that all elements of 1st column of the array is positive but the function element$(1 - x_1^2)$ of DRA is indefinite so systems' stability is not guaranteed for all values of x_1.

The system's stability is guaranteed when the trajectory of the system crosses the boundary of x_1(position axis) and converse at the equilibrium point but, if trajectory doesn't cross the boundary of x_1(position axis) and diverse towards infinity then system is unstable.

So, the system is stable for ($x_1 < 1$) as shown in Fig.11. The trajectory of the system is showing (Fig.10) that system is stable for $x_1 < 1$ and system becomes unstable when$x_1 > 1$. The DPs are moving on the

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LHS of g-plane when $x_1 < 1$, the system is stable but when the value of x_1 increase and then the DP move from LHS to RHS of g-plane, the system becomes unstable for $x_1 < 1$ as shown in Fig.11. The system stability is guaranteed when the trajectory enters into $((1 - x_1^2) > 0)$ region otherwise system is unstable.

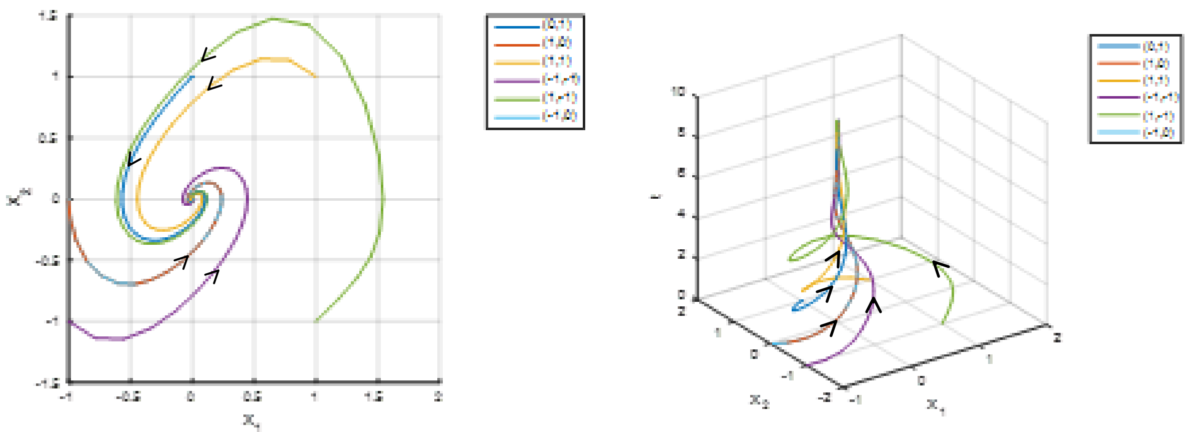
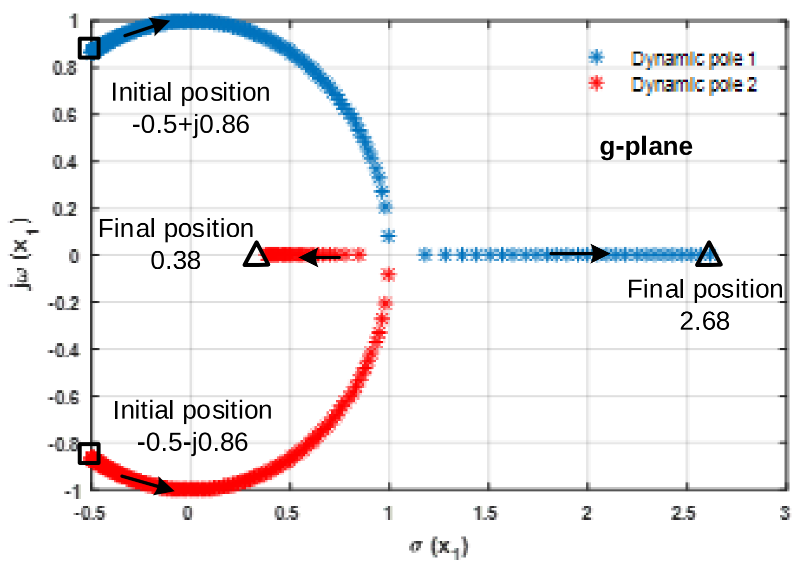


Figure 10. Trajectories in phase plane (x_1, x_2) (a) 2-D (b) 3-D



(a)

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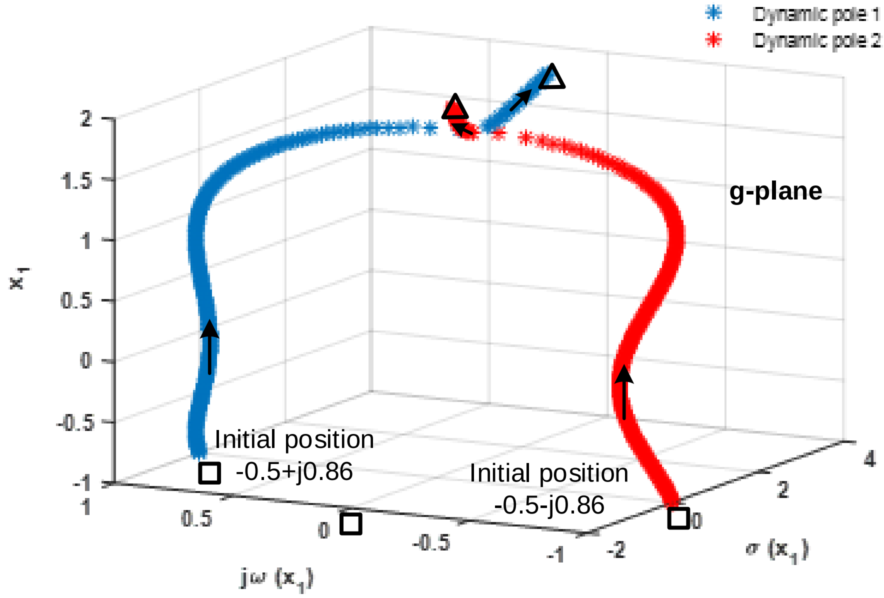

(b)

Figure 11. Dynamic Root Locus for Eqn.32 showing dynamic pole motion approach for stability (a) 2-D (b) 3-D

4. Conclusions:

The nonlinearity of dynamic systems and their corresponding stability characteristics have been thoroughly analyzed through simulations. A stability criterion based on the Dynamic-Routh method, utilizing the dominant pole (DP) motion approach, has been successfully developed for nonlinear dynamic systems. The effectiveness and accuracy of this method have been validated through several numerical examples involving nonlinear system equations. These case studies demonstrate that the Dynamic-Routh stability approach not only offers reliable results but also proves to be more efficient and straightforward compared to traditional stability analysis methods. As a result, it stands out as a practical alternative when evaluating the stability of nonlinear systems.

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