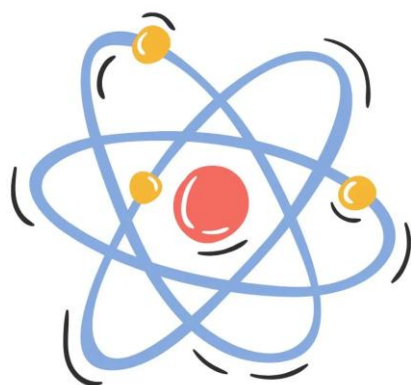




ISSN: 1672 - 6553

JOURNAL OF DYNAMICS AND CONTROL

VOLUME 9 ISSUE 5: 57 - 73

SECOND ZAGREB INDEX OF k - SPLITTING GRAPH OF GENERALIZED TRANSFORMATION GRAPHS

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Abstract: Recently, in [18] Raikar et al. obtained explicit formulae for the first Zagreb index and coindex of k -splitting of generalized transformation graphs and their complement. The aim of this paper is to obtain explicit formulae for the second Zagreb index and coindex of k -splitting of generalized transformation graphs $spl_k(G^{ab})$, and followed by this we also obtained analogous expressions for the complements of $spl_k(G^{ab})$.

2010 Mathematics Subject Classifications: 05C05, 05C07, 05C35.

1 Introduction

The vertices of molecular graph correspond to the atoms in the molecule and the edges correspond to the covalent bonds. A topological index is a graph invariant which maps each molecular graph to a real number. Numerous such descriptors have been considered in theoretical chemistry, and have found some applications, especially in QSPR/QSAR [14, 21]. Topological indices are categorized into two types namely degree-based indices and distance-based indices. Some of the well known degree-based indices are first Zagreb index, second Zagreb index, forgotten index, hyper Zagreb index, Randic index, harmonic index, geometric-arithmetic index, redefined third Zagreb index, etc. For more details on degree based topological indices we refer to [6, 7, 8, 9, 10].

Among all these degree-based indices the first and second Zagreb indices are studied widely. In 1972, Gutman et al. [7] defined the first and second Zagreb indices of a graph G . Till today, many researchers over the globe are exploring these indices to the advanced levels. Later in 2008, Došlić [5] defined the first and second Zagreb coindices, due to all non-adjacent pair of vertices. For more information on the Zagreb indices and coindices with their applications see [2, 10, 15, 16, 17, 20, 24]. In 2015, Basavanagoud et al. [3] defined new graph operations called generalized transformation G^{ab} , and obtained the expressions for the first and second Zagreb indices and coindices of these graphs and their complements. In 2017, Vaidya et al. [23] defined a new graph operation called k -splitting of a graph G and studied energy of the same.

Recently, in [18] Raikar et al. obtained explicit formulae for the first Zagreb index and coindex of k -splitting of generalized transformation graphs and their complement. Motivated by this, in the present work we focus on obtaining explicit formulae for the second Zagreb index and coindex of k -splitting of generalized transformation graphs $spl_k(G^{ab})$, and followed by this we also obtain analogous expressions for the complements of $spl_k(G^{ab})$.

2 Preliminaries

Throughout this paper, we consider undirected, simple and finite graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$ such that $|V(G)| = n$ and $|E(G)| = m$, where members of $V(G)$ are called vertices and members of $E(G)$ are called edges of graph G . Two vertices are said to be adjacent in G if they share a common edge and this edge is said to be incident on those two vertices. The degree of a vertex u in $V(G)$, denoted by $d_G(u)$, is the number of edges which are incident with u . The complement $\overline{G}$ of the graph G is a simple graph on the same vertex set of G such that two vertices u and v are adjacent in $\overline{G}$ if and only if they are non-adjacent in G . For other graph theoretic definitions, we refer the book [13].

The first Zagreb index $M_1(G)$ [7] and second Zagreb index $M_2(G)$ [7] are defined as

$$M_1(G) = \sum_{u \in V(G)} d_G(u)^2 \quad \text{and} \quad M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v)$$

respectively.

The first Zagreb can also be expressed as [4]

$$M_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)].$$

The first Zagreb coindex $\overline{M}_1(G)$ [5] and second Zagreb coindex $\overline{M}_2(G)$ [5] are defined as

$$\overline{M}_1(G) = \sum_{uv \notin E(G)} [d_G(u) + d_G(v)] \quad \text{and} \quad \overline{M}_2(G) = \sum_{uv \notin E(G)} d_G(u) d_G(v)$$

respectively.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and let α, β be two elements of $V(G) \cup E(G)$. We say that the associativity of α and β is $+$ if they are adjacent or incident in G , otherwise is $-$. Let ab be a 2-permutation of the set $\{+, -\}$. We say that α and β correspond to the first term a of ab if both α and β are in $V(G)$, whereas α and β correspond to the second term b of ab if one of α and β is in $V(G)$ and the other is in $E(G)$. The generalized transformation graph G^{ab} of G is defined on the vertex set $V(G) \cup E(G)$. Two vertices α and β of G^{ab} are joined by an edge if and only if their associativity in G is consistent with the corresponding term of ab . With respect to this, there are four graphical transformations of graphs such as $G^{++}, G^{+-}, G^{-+}, G^{--}$, as there are four distinct 2-permutations of $\{+, -\}$.

In another way, the generalized transformation graph G^{ab} is a graph whose vertex set is $V(G) \cup E(G)$, and $\alpha, \beta \in V(G^{ab})$ such that α and β are adjacent in G^{ab} if and only if (i) and (ii) holds:

- (i) $\alpha, \beta \in V(G^{ab})$, α, β are adjacent in G if $a = +$ and α, β are not adjacent in G if $a = -$.
- (ii) $\alpha \in V(G)$ and $\beta \in E(G)$, α, β are incident in G if $b = +$ and α, β are not incident in G if $b = -$.

The vertex u of G^{ab} corresponding to a vertex u of G is referred to as a point vertex. The vertex e of G^{ab} corresponding to an edge e of G is referred to as a line vertex.

The k -splitting of a graph G , denoted by $spl_k(G)$, is obtained by adding to each vertex u of G new k vertices, say $u', u'', \dots, u^{(k)}$ such that $u^{(i)}$, $1 \leq i \leq k$, is adjacent to each vertex that is adjacent to u in G .

The following results are required for the present considerations.

Lemma 2.1 [1, 10] For a graph G with n vertices and m edges

$$M_1(\overline{G}) = M_1(G) + n(n-1)^2 - 4m(n-1).$$

Lemma 2.2 [10, 20] For a graph G with n vertices and m edges

$$\overline{M}_1(G) = 2m(n-1) - M_1(G).$$

Lemma 2.3 [10, 20] For a graph G with n vertices and m edges

$$\overline{M}_1(\overline{G}) = \overline{M}_1(G).$$

Lemma 2.4 [10, 20] For a graph G with n vertices and m edges

$$M_2(\overline{G}) = \frac{1}{2}n(n-1)^3 - 3m(n-1)^2 + 2m^2 + \frac{2n-3}{2}M_1(G) - M_2(G).$$

Lemma 2.5 [10, 20] For a graph G with n vertices and m edges

$$\overline{M}_2(G) = 2m^2 - \frac{1}{2}M_1(G) - M_2(G).$$

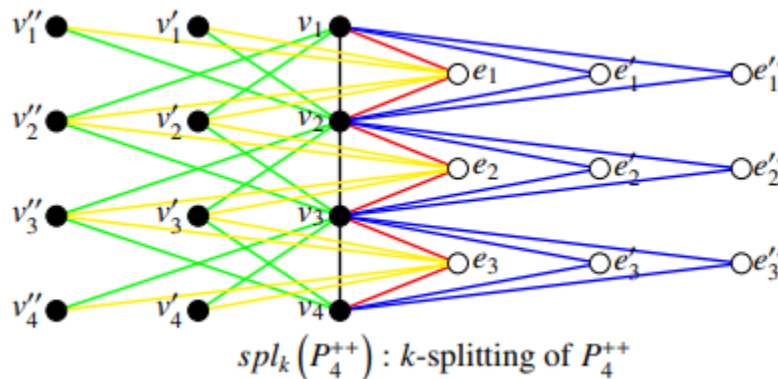
Lemma 2.6 [10, 20] For a graph G with n vertices and m edges

$$\overline{M}_2(\overline{G}) = m(n-1)^2 - (n-1)M_1(G) + M_2(G).$$

Now we present the main results of our work through following sections.

3 Second Zagreb Index of $spl_k(G^{++})$

Let G be a (n, m) -graph and $spl_k(G^{++})$ represents k -splittig of G^{++} .



Proposition 3.1 [18] Let G be a (n, m) -graph. Then

$$(i) \quad d_{spl_k(G^{++})}(v) = 2(k+1)d_G(v) \quad \text{where } v \in V(G)$$

$$(ii) \ d_{spl_k(G^{++})}(e) = 2(k + 1) \text{ where } e \in E(G)$$

$$(iii) \ d_{spl_k(G^{++})}(v') = 2d_G(v) \text{ where } v' \text{ is vertex in } k - \text{splitting of } G^{++} \text{ due to vertex } v \text{ in } G$$

$$(iv) \ d_{spl_k(G^{++})}(e') = 2 \text{ where } e' \text{ is vertex in } k - \text{splitting of } G^{++} \text{ due to edge } e \text{ in } G.$$

Proposition 3.2 [18] *Let G be a (n, m) -graph. Then order and size of $spl_k(G^{++})$ are $(n + m)(k + 1)$ and $3m(k + 1)$.*

Proposition 3.3 [18] *Let G be a (n, m) -graph. Then*

$$M_1(spl_k(G^{++})) = 4(k + (k + 1)^2)(M_1(G) + m).$$

Theorem 3.4 *Let G be a (n, m) -graph. Then*

$$M_2(spl_k(G^{++})) = 4(k + 1)(3k + 1)(M_2(G) + M_1(G)).$$

Proof. Partitioning edge set of $spl_k(G^{++})$ as follows:

$$E(spl_k(G^{++})) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$$

where

$$E_1 = \{uv: uv \in E(G)\}$$

$$E_2 = \{ue: \text{vertex } u \text{ in } G \text{ is incident to edge } e \text{ in } G\}$$

$$E_3 = \{uv': \text{vertex } u \text{ in } G \text{ is adjacent to vertex } v' \text{ in } k - \text{splitting of } G^{++} \text{ due to vertex } v \text{ in } G\}$$

$$E_4 = \{ue': \text{vertex } u \text{ in } G \text{ is incident to vertex } e' \text{ in } k - \text{splitting of } G^{++} \text{ due to edge } e \text{ in } G\}$$

$$E_5 = \{u'e: \text{vertex } u' \text{ in } k - \text{splitting of } G^{++} \text{ due to vertex } u \text{ in } G \text{ incident to edge } e \text{ in } G\}$$

Clearly $|E_1| = m, |E_2| = 2m, |E_3| = 2mk, |E_4| = 2mk$ and $|E_5| = 2mk$
Consider,

$$\begin{aligned} M_2(spl_k(G^{++})) &= \sum_{uv \in (E(spl_k(G^{++})))} d_{spl_k(G^{++})}(u) d_{spl_k(G^{++})}(v) \\ &= \sum_{uv \in E_1} d_{spl_k(G^{++})}(u) d_{spl_k(G^{++})}(v) + \sum_{ue \in E_2} d_{spl_k(G^{++})}(u) d_{spl_k(G^{++})}(e) \\ &+ \sum_{uv' \in E_3} d_{spl_k(G^{++})}(u) d_{spl_k(G^{++})}(v') + \sum_{ue' \in E_4} d_{spl_k(G^{++})}(u) d_{spl_k(G^{++})}(e') \end{aligned}$$

$$\begin{aligned}
 & + \sum_{u'e \in E_5} d_{spl_k(G^{++})}(u') d_{spl_k(G^{++})}(e) \\
 & = \sum_{uv \in E_1} 2(k+1)d_G(u)2(k+1)d_G(v) + \sum_{ue \in E_2} 2(k+1)d_G(u)2(k+1) \\
 & + \sum_{uv' \in E_3} 2(k+1)d_G(u)2d_G(v) + \sum_{ue' \in E_4} 2(k+1)d_G(u)2 + \sum_{u'e \in E_5} 2d_G(u)2(k+1) \\
 & = 4(k+1)^2 \sum_{uv \in E_1} d_G(u)d_G(v) + 4(k+1)^2 \sum_{ue \in E_2} d_G(u) \\
 & + 4(k+1) \sum_{uv' \in E_3} d_G(u)d_G(v) + 4(k+1) \sum_{ue' \in E_4} d_G(u) + 4(k+1) \sum_{u'e \in E_5} d_G(u) \\
 & = 4(k+1)^2 \sum_{uv \in E(G)} d_G(u)d_G(v) + 4(k+1)^2 \sum_{u \in V(G)} d_G(u)^2 \\
 & + 4(k+1) 2k \sum_{uv \in E(G)} d_G(u)d_G(v) + 4(k+1) k \sum_{u \in E(G)} d_G(u)^2 + 4(k+1) k \sum_{u \in E(G)} d_G(u)^2 \\
 & = 4(k+1)^2 M_2(G) + 4(k+1)^2 M_1(G) + 8k(k+1)M_2(G) + 4k(k+1)M_1(G) \\
 & + 4k(k+1)M_1(G) \\
 & = 4(k+1)^2 (M_2(G) + M_1(G)) + 8k(k+1)(M_2(G) + M_1(G)) \\
 & = (4(k+1)^2 + 8k(k+1))(M_2(G) + M_1(G)) \\
 & = 4(k+1)((k+1) + 2k)(M_2(G) + M_1(G)) \\
 & = 4(k+1)(3k+1)(M_2(G) + M_1(G)).
 \end{aligned}$$

Corollary 3.5 Let G be a (n, m) -graph and $spl_k(G^{++})$ be k -splitting graph of G^{++} . Then

$$\overline{M_2}(spl_k(G^{++})) = \frac{1}{2}n'(n'-1)^3 - 3m'(n'-1)^2 + 2m'^2 + \frac{2n'-3}{2}M_1(spl_k(G^{++})) - M_2(spl_k(G^{++}))$$

where $n' = (n + m)$ and $m' = 3m(k + 1)$.

Proof. Result follows from Lemma 2.4.

Corollary 3.6 Let G be a (n, m) -graph and $spl_k(G^{++})$ be k -splitting graph of G^{++} . Then

$$\overline{M_2}(spl_k(G^{++})) = 2m'^2 - \frac{1}{2}M_1(spl_k(G^{++})) - M_2(spl_k(G^{++}))$$

where $n' = (n + m)$ and $m' = 3m(k + 1)$.

Proof. Result follows from Lemma 2.5.

Corollary 3.7 Let G be a (n, m) -graph and $spl_k(G^{++})$ be k -splitting graph of G^{++} . Then

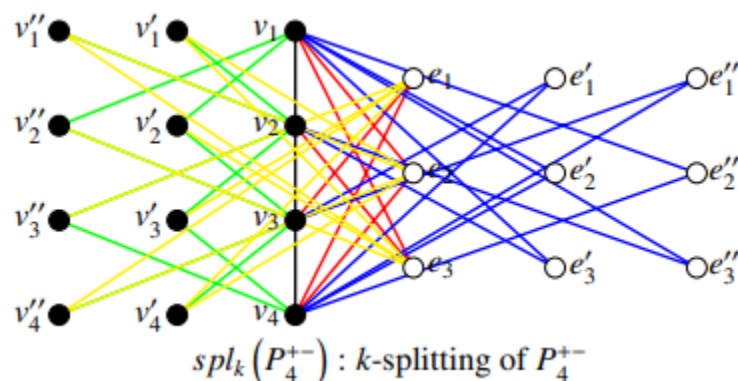
$$\overline{M_2}(spl_k(G^{++})) = m'(n'-1)^2 - (n'-1)M_1(spl_k(G^{++})) + M_2(spl_k(G^{++}))$$

where $n' = (n + m)$ and $m' = 3m(k + 1)$.

Proof. Result follows from Lemma 2.6.

4 Second Zagreb Index of $spl_k(G^{+-})$

Let G be a (n, m) -graph and $spl_k(G^{+-})$ represents k -splitting of G^{+-} .



Proposition 4.1 [18] Let G be a (n, m) -graph. Then

- (i) $d_{spl_k(G^{+-})}(v) = m(k + 1)$ where $v \in V(G)$
- (ii) $d_{spl_k(G^{+-})}(e) = (n - 2)(k + 1)$ where $e \in E(G)$
- (iii) $d_{spl_k(G^{+-})}(v') = m$ where v' is vertex in k -splitting of G^{+-} due to vertex v in G
- (iv) $d_{spl_k(G^{+-})}(e') = (n - 2)$ where e' is vertex in k -splitting of G^{+-} due to edge e in G .

Proposition 4.2 [18] Let G be a (n, m) -graph. Then order and size of $spl_k(G^{+-})$ are $(n + m)(k + 1)$ and $m(n - 1)(2k + 1)$.

Proposition 4.3 [18] Let G be a (n, m) -graph. Then

$$M_1(spl_k(G^{+-})) = m(mn + (n - 2)^2)(k + (k + 1)^2).$$

Theorem 4.4 Let G be a (n, m) -graph. Then

$$M_2(spl_k(G^{+-})) = m^2(k + 1)(3k + 1)(m + (n - 2)^2).$$

Proof. Partitioning edge set of $spl_k(G^{+-})$ as follows:

$$E(spl_k(G^{+-})) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$$

where

$$E_1 = \{uv : uv \in E(G)\}$$

$E_2 = \{ue: \text{vertex } u \text{ in } G \text{ is not incident to edge } e \text{ in } G\}$
 $E_3 = \{uv': \text{vertex } u \text{ in } G \text{ is adjacent to vertex } v' \text{ in } k -$
 splitting of G^{+-} due to vertex v in $G\}$
 $E_4 = \{ue': \text{vertex } u \text{ in } G \text{ is incident to vertex } e' \text{ in } k -$
 splitting of G^{+-} due to edge e in $G\}$
 $E_5 = \{u'e: \text{vertex } u' \text{ in } k -$
 splitting of G^{+-} due to vertex u in G not incident to edge e in $G\}$

Clearly, $|E_1| = m$, $|E_2| = m(n - 2)$, $|E_3| = 2mk$, $|E_4| = mk(n - 2)$ and $|E_5| = mk(n - 2)$.

Consider,

$$\begin{aligned}
 M_2(spl_k(G^{+-})) &= \sum_{uv \in (E(spl_k(G^{+-})))} d_{spl_k(G^{+-})}(u) d_{spl_k(G^{+-})}(v) \\
 &= \sum_{uv \in E_1} d_{spl_k(G^{+-})}(u) d_{spl_k(G^{+-})}(v) + \sum_{ue \in E_2} d_{spl_k(G^{+-})}(u) d_{spl_k(G^{+-})}(e) \\
 &+ \sum_{uv' \in E_3} d_{spl_k(G^{+-})}(u) d_{spl_k(G^{+-})}(v') + \sum_{ue' \in E_4} d_{spl_k(G^{+-})}(u) d_{spl_k(G^{+-})}(e') \\
 &+ \sum_{u'e \in E_5} d_{spl_k(G^{+-})}(u') d_{spl_k(G^{+-})}(e) \\
 &= \sum_{uv \in E_1} m(k+1) m(k+1) + \sum_{ue \in E_2} m(k+1) (n-2)(k+1) + \sum_{uv' \in E_3} m(k+1) m \\
 &+ \sum_{ue' \in E_4} m(k+1) (n-2) + \sum_{u'e \in E_5} m (n-2)(k+1) \\
 &= m^2(k+1)^2 \sum_{uv \in E_1} 1 + m(n-2)(k+1)^2 \sum_{ue \in E_2} 1 + m^2(k+1) \sum_{uv' \in E_3} 1 \\
 &+ m(n-2)(k+1) \sum_{ue' \in E_4} 1 + m(n-2)(k+1) \sum_{u'e \in E_5} 1 \\
 &= m^2(k+1)^2 m + m(n-2)(k+1)^2 m(n-2) + m^2(k+1) 2mk \\
 &+ m(n-2)(k+1) mk(n-2) + m(n-2)(k+1) mk(n-2) \\
 &= m^3(k+1)^2 + m^2(n-2)^2(k+1)^2 + 2m^3k(k+1) + 2m^2(n-2)^2k(k+1) \\
 &= (m^3 + m^2(n-2)^2)(k+1)^2 + (m^3 + m^2(n-2)^2)2k(k+1) \\
 &= (m^3 + m^2(n-2)^2)((k+1)^2 + 2k(k+1)) \\
 &= (k+1)(3k+1)(m^3 + m^2(n-2)^2) \\
 &= m^2(k+1)(3k+1)(m + (n-2)^2).
 \end{aligned}$$

Corollary 4.5 Let G be a (n, m) -graph and $spl_k(G^{+-})$ be k -splitting graph of G^{+-} . Then

$$M_2(\overline{spl_k(G^{+-})}) = \frac{1}{2}n'(n'-1)^3 - 3m'(n'-1)^2 + 2m'^2 + \frac{2n'-3}{2}M_1(spl_k(G^{+-})) - M_2(spl_k(G^{+-}))$$

where $n' = (n + m)$ and $m' = m(n - 1)(2k + 1)$.

Proof. Result follows from Lemma 2.4.

Corollary 4.6 Let G be a (n, m) -graph and $spl_k(G^{+-})$ be k -splitting graph of G^{+-} . Then

$$\overline{M}_2(spl_k(G^{+-})) = 2m'^2 - \frac{1}{2}M_1(spl_k(G^{+-})) - M_2(spl_k(G^{+-}))$$

where $n' = (n + m)$ and $m' = m(n - 1)(2k + 1)$.

Proof. Result follows from Lemma 2.5.

Corollary 4.7 Let G be a (n, m) -graph and $spl_k(G^{+-})$ be k -splitting graph of G^{+-} . Then

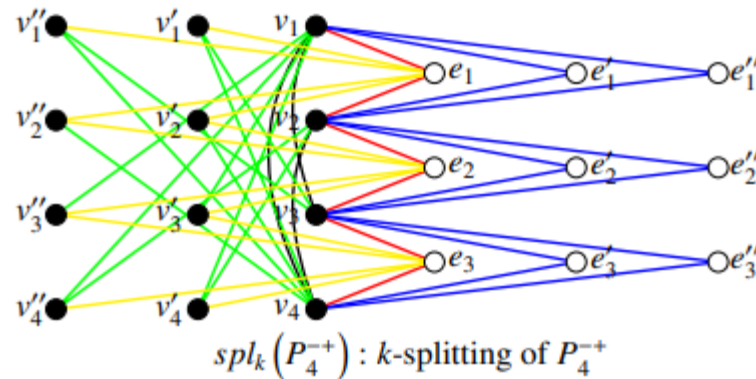
$$\overline{M}_2(spl_k(G^{+-})) = m'(n' - 1)^2 - (n' - 1)M_1(spl_k(G^{+-})) + M_2(spl_k(G^{+-}))$$

where $n' = (n + m)$ and $m' = m(n - 1)(2k + 1)$.

Proof. Result follows from Lemma 2.6.

5 Second Zagreb Index of $spl_k(G^{-+})$

Let G be a (n, m) -graph and $spl_k(G^{-+})$ represents k -splitting of G^{-+} .



Proposition 5.1 [18] Let G be a (n, m) -graph. Then

- (i) $d_{spl_k(G^{-+})}(v) = (n - 1)(k + 1)$ where $v \in V(G)$
- (ii) $d_{spl_k(G^{-+})}(e) = 2(k + 1)$ where $e \in E(G)$
- (iii) $d_{spl_k(G^{-+})}(v') = (n - 1)$ where v' is vertex in k -splitting of G^{-+} due to vertex v in G
- (iv) $d_{spl_k(G^{-+})}(e') = 2$ where e' is vertex in k -splitting of G^{-+} due to edge e in G .

Proposition 5.2 [18] Let G be a (n, m) -graph. Then order and size of $spl_k(G^{-+})$ are $(n + m)(k + 1)$ and $\frac{1}{2}[n(n - 1) + 2m](2k + 1)$.

Proposition 5.3 [18] Let G be a (n, m) -graph. Then $M_1(spl_k(G^{-+})) = (n(n - 1)^2 + 4m)(k + (k + 1)^2)$.

Theorem 5.4 Let G be a (n, m) -graph. Then

$$M_2(spl_k(G^{-+})) = (n - 1)(k + 1)(3k + 1) \left((n - 1) \left(\frac{n(n - 1)}{2} - m \right) + 4m \right).$$

Proof. Partitioning edge set of $spl_k(G^{-+})$ as follows:

$$E(spl_k(G^{-+})) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$$

where

$$E_1 = \{uv: uv \notin E(G)\}$$

$$E_2 = \{ue: \text{vertex } u \text{ in } G \text{ is incident to edge } e \text{ in } G\}$$

$$E_3 = \{uv': \text{vertex } u \text{ in } G \text{ is not adjacent to vertex } v' \text{ in } k - \text{splitting of } G^{-+} \text{ due to vertex } v \text{ in } G\}$$

$$E_4 = \{ue': \text{vertex } u \text{ in } G \text{ is incident to vertex } e' \text{ in } k - \text{splitting of } G^{-+} \text{ due to edge } e \text{ in } G\}$$

$$E_5 = \{u'e: \text{vertex } u' \text{ in } k - \text{splitting of } G^{-+} \text{ due to vertex } u \text{ in } G \text{ is not incident to edge } e \text{ in } G\}$$

Clearly, $|E_1| = \frac{n(n-1)}{2} - m$, $|E_2| = 2m$, $|E_3| = 2k \left(\frac{n(n-1)}{2} - m \right)$, $|E_4| = 2mk$ and $|E_5| = 2mk$.

Consider,

$$\begin{aligned} M_2(spl_k(G^{-+})) &= \sum_{uv \in (E(spl_k(G^{-+})))} d_{spl_k(G^{-+})}(u) d_{spl_k(G^{-+})}(v) \\ &= \sum_{uv \in E_1} d_{spl_k(G^{-+})}(u) d_{spl_k(G^{-+})}(v) + \sum_{ue \in E_2} d_{spl_k(G^{-+})}(u) d_{spl_k(G^{-+})}(e) \\ &+ \sum_{uv' \in E_3} d_{spl_k(G^{-+})}(u) d_{spl_k(G^{-+})}(v') + \sum_{ue' \in E_4} d_{spl_k(G^{-+})}(u) d_{spl_k(G^{-+})}(e') \\ &+ \sum_{u'e \in E_5} d_{spl_k(G^{-+})}(u') d_{spl_k(G^{-+})}(e) \\ &= \sum_{uv \in E_1} (n - 1)(k + 1)(n - 1)(k + 1) + \sum_{ue \in E_2} (n - 1)(k + 1)2(k + 1) \end{aligned}$$

$$\begin{aligned}
 & + \sum_{uv' \in E_3} (n-1)(k+1)(n-1) + \sum_{ue' \in E_4} (n-1)(k+1)2 + \sum_{ue \in E_5} (n-1)2(k+1) \\
 & = (n-1)^2(k+1)^2 \sum_{uv \in E_1} 1 + 2(n-1)(k+1)^2 \sum_{ue \in E_2} 1 + (n-1)^2(k+1) \sum_{uv' \in E_3} 1 \\
 & \quad + 2(n-1)(k+1) \sum_{ue' \in E_4} 1 + 2(n-1)(k+1) \sum_{ue \in E_5} 1 \\
 & = (n-1)^2(k+1)^2 m' + 2(n-1)(k+1)^2 2m + (n-1)^2(k+1)2km' \\
 & \quad + 2(n-1)(k+1)2mk + 2(n-1)(k+1)2mk \\
 & = (n-1)^2(k+1)((k+1) + 2k)m' + 4m(n-1)(k+1)((k+1) + 2k) \\
 & = ((n-1)^2(k+1)m' + 4m(n-1)(k+1))(3k+1) \\
 & = (n-1)(k+1)(3k+1)((n-1)m' + 4m) \\
 & = (n-1)(k+1)(3k+1) \left((n-1) \left(\frac{n(n-1)}{2} - m \right) + 4m \right).
 \end{aligned}$$

Corollary 5.5 Let G be a (n, m) -graph and $spl_k(G - +)$ be k -splitting graph of G^{-+} . Then

$$\overline{M_2}(spl_k(G^{-+})) = \frac{1}{2}n'(n'-1)^3 - 3m'(n'-1)^2 + 2m'^2 + \frac{2n'-3}{2}M_1(spl_k(G^{-+})) - M_2(spl_k(G^{-+}))$$

where $n' = (n + m)$ and $m' = \frac{1}{2}[n(n-1) + 2m](2k+1)$.

Proof. Result follows from Lemma 2.4.

Corollary 5.6 Let G be a (n, m) -graph and $spl_k(G - +)$ be k -splitting graph of G^{-+} . Then

$$\overline{M_2}(spl_k(G^{-+})) = 2m'^2 - \frac{1}{2}M_1(spl_k(G^{-+})) - M_2(spl_k(G^{-+}))$$

where $n' = (n + m)$ and $m' = \frac{1}{2}[n(n-1) + 2m](2k+1)$.

Proof. Result follows from Lemma 2.5.

Corollary 5.7 Let G be a (n, m) -graph and $spl_k(G - +)$ be k -splitting graph of G^{-+} . Then

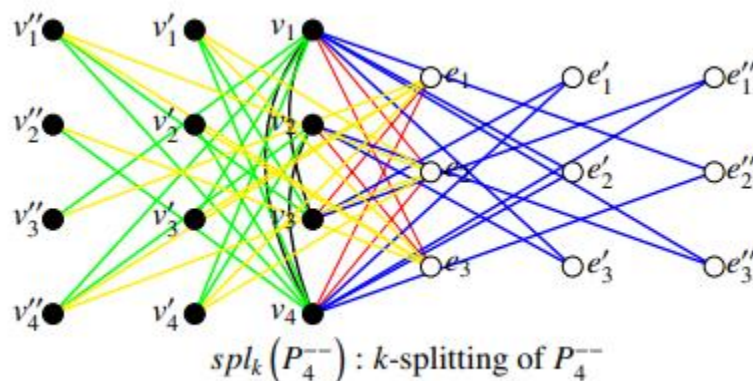
$$\overline{M_2}(spl_k(G^{-+})) = m'(n'-1)^2 - (n'-1)M_1(spl_k(G^{-+})) + M_2(spl_k(G^{-+}))$$

where $n' = (n + m)$ and $m' = \frac{1}{2}[n(n-1) + 2m](2k+1)$.

Proof. Result follows from Lemma 2.6.

6 Second Zagreb Index of $spl_k(G^{--})$

Let G be a (n, m) -graph and $spl_k(G^{--})$ represents k -splittig of G^{--} .



Proposition 6.1 [18] Let G be a (n, m) -graph. Then

- (i) $d_{spl_k(G^{--})}(v) = (n + m - 1 - 2d_G(v))(k + 1)$ where $v \in V(G)$
- (ii) $d_{spl_k(G^{--})}(e) = (n - 2)(k + 1)$ where $e \in E(G)$
- (iii) $d_{spl_k(G^{--})}(v') = (n + m - 1 - 2d_G(v))$ where v' is vertex in k -splittig of G^{--} due to vertex v in G
- (iv) $d_{spl_k(G^{--})}(e') = (n - 2)$ where e' is vertex in k -splittig of G^{--} due to edge e in G .

Proposition 6.2 [18] Let G be a (n, m) -graph. Then order and size of $spl_k(G^{--})$ are $(n + m)(k + 1)$ and

$$\frac{1}{2}[n(n - 1) + 2m(n - 3)](2k + 1).$$

Proposition 6.3 [18] Let G be a (n, m) -graph. Then

$$M_1(spl_k(G^{--})) = (n(n - 1)^2 + m^2(n - 4) + m + 3m(n - 1)(n - 5) + 4M_1(G))(k + (k + 1)^2).$$

Theorem 6.4 Let G be a (n, m) -graph. Then

$$M_2(spl_k(G^{--})) = (k + 1)(3k + 1)[(n + m - 1)^2 \left(\frac{n(n-1)}{2} - m \right) - 2(n + m - 1)\overline{M}_1(G) + 4\overline{M}_2(G) + (n + m - 1)m(n - 2) - 2(n - 1)2m + 2M_1(G)].$$

Proof. Partitioning edge set of $spl_k(G^{--})$ as follows:

$$E(spl_k(G^{--})) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$$

where

$$\begin{aligned}
 E_1 &= \{uv: uv \notin E(G)\} \\
 E_2 &= \{ue: \text{vertex } u \text{ in } G \text{ is not incident to edge } e \text{ in } G\} \\
 E_3 &= \{uv': \text{vertex } u \text{ in } G \text{ is not adjacent to vertex } v' \text{ in } k - \\
 &\text{splitting of } G^{--} \text{ due to vertex } v \text{ in } G\} \\
 E_4 &= \{ue': \text{vertex } u \text{ in } G \text{ is not incident to vertex } e' \text{ in } k - \\
 &\text{splitting of } G^{--} \text{ due to edge } e \text{ in } G\} \\
 E_5 &= \{u'e: \text{vertex } u' \text{ in } k - \\
 &\text{splitting of } G^{--} \text{ due to vertex } u \text{ in } G \text{ is not incident to edge } e \text{ in } G\}
 \end{aligned}$$

Clearly, $|E_1| = \frac{n(n-1)}{2} - m$, $|E_2| = m(n-2)$, $|E_3| = 2k \left(\frac{n(n-1)}{2} - m \right)$, $|E_4| = mk(n-2)$ and $|E_5| = mk(n-2)$.

Consider,

$$\begin{aligned}
 M_2(spl_k(G^{--})) &= \sum_{uv \in (E(spl_k(G^{--})))} d_{spl_k(G^{--})}(u) d_{spl_k(G^{--})}(v) \\
 &= \sum_{uv \in E_1} d_{spl_k(G^{--})}(u) d_{spl_k(G^{--})}(v) + \sum_{ue \in E_2} d_{spl_k(G^{--})}(u) d_{spl_k(G^{--})}(e) \\
 &+ \sum_{uv' \in E_3} d_{spl_k(G^{--})}(u) d_{spl_k(G^{--})}(v') + \sum_{ue' \in E_4} d_{spl_k(G^{--})}(u) d_{spl_k(G^{--})}(e') \\
 &\quad + \sum_{u'e \in E_5} d_{spl_k(G^{--})}(u') d_{spl_k(G^{--})}(e) \\
 &= \sum_{uv \in E_1} (n+m-1-2d_G(u))(k+1)(n+m-1-2d_G(v))(k+1) \\
 &\quad + \sum_{ue \in E_2} (n+m-1-2d_G(u))(n-2)(k+1) \\
 &\quad + \sum_{uv' \in E_3} (n+m-1-2d_G(u))(k+1)(n+m-1-2d_G(v)) \\
 &+ \sum_{ue' \in E_4} (n+m-1-2d_G(u))(k+1)(n-2) + \sum_{u'e \in E_5} (n+m-1-2d_G(u))(n-2)(k \\
 &\quad + 1) \\
 &= (k+1)^2 \sum_{uv \in E_1} [(n+m-1)^2 - 2(n+m-1)(d_G(u) + d_G(v)) + 4d_G(u)d_G(v)] \\
 &\quad + (n-2)(k+1)^2 \sum_{ue \in E_2} (n+m-1-2d_G(u)) \\
 &+ (k+1) \sum_{uv' \in E_3} [(n+m-1)^2 - 2(n+m-1)(d_G(u) + d_G(v)) + 4d_G(u)d_G(v)] \\
 &\quad + (n-2)(k+1) \sum_{ue' \in E_4} (n+m-1-2d_G(u)) + (n-2)(k \\
 &\quad + 1) \sum_{u'e \in E_5} (n+m-1-2d_G(u))
 \end{aligned}$$

$$\begin{aligned}
 &= (k+1)^2 \left[(n+m-1)^2 \sum_{uv \in E_1} 1 - 2(n+m-1) \sum_{uv \in E_1} (d_G(u) + d_G(v)) \right. \\
 &\quad \left. + 4 \sum_{uv \in E_1} d_G(u) d_G(v) \right] \\
 &\quad + (n-2)(k+1)^2 \left[(n+m-1) \sum_{ue \in E_2} 1 - 2 \sum_{ue \in E_2} d_G(u) \right] \\
 &\quad + (k+1) \left[(n+m-1)^2 \sum_{uv \in E_3} 1 - 2(n+m-1) \sum_{uv \in E_3} (d_G(u) + d_G(v)) \right. \\
 &\quad \left. + 4 \sum_{uv \in E_3} d_G(u) d_G(v) \right] \\
 &\quad + (n-2)(k+1) \left[(n+m-1) \sum_{ue \in E_4} 1 - 2 \sum_{ue \in E_4} d_G(u) \right] \\
 &\quad + (n-2)(k+1) \left[(n+m-1) \sum_{ue \in E_5} 1 - 2 \sum_{ue \in E_5} d_G(u) \right]
 \end{aligned}$$

$$\begin{aligned}
 M_2(spl_k(G^{--})) &= (k+1)^2 \left[(n+m-1)^2 \sum_{uv \notin E(G)} 1 - 2(n+m-1) \sum_{uv \notin E(G)} (d_G(u) + d_G(v)) \right. \\
 &\quad \left. + 4 \sum_{uv \notin E(G)} d_G(u) d_G(v) \right] \\
 &\quad + (n-2)(k+1)^2 \left[(n+m-1) \sum_{ue \in E_2} 1 - 2 \sum_{u \in V(G)} (n-1-d_G(u)) d_G(u) \right] \\
 &\quad + (k+1) \left[(n+m-1)^2 \sum_{uv \in E_3} 1 - 2(n+m-1)(2k) \sum_{uv \notin E(G)} (d_G(u) + d_G(v)) \right. \\
 &\quad \left. + 4(2k) \sum_{uv \notin E(G)} d_G(u) d_G(v) \right] \\
 &\quad + (n-2)(k+1) \left[(n+m-1) \sum_{ue \in E_4} 1 - 2k \sum_{u \in V(G)} (n-1-d_G(u)) d_G(u) \right]
 \end{aligned}$$

$$\begin{aligned}
 & + (n-2)(k+1) \left[(n+m-1) \sum_{u \in E_g} 1 - 2k \sum_{u \in V(G)} (n-1-d_G(u))d_G(u) \right] \\
 & = (k+1)^2 \left[(n+m-1)^2 \left(\frac{n(n-1)}{2} - m \right) - 2(n+m-1)\overline{M}_1(G) + 4\overline{M}_2(G) \right] \\
 & \quad + (n-2)(k+1)^2 [(n+m-1)m(n-2) - 2(n-1)2m + 2M_1(G)] \\
 & \quad + (k+1) \left[(n+m-1)2k \left(\frac{n(n-1)}{2} - m \right) - 4k(n+m-1)\overline{M}_1(G) + 8k\overline{M}_2(G) \right] \\
 & \quad + (n-2)(k+1) [(n+m-1)mk(n-2) - 2k(n-1)2m + 2kM_1(G)] \\
 & \quad + (n-2)(k+1) [(n+m-1)mk(n-2) - 2k(n-1)2m + 2kM_1(G)] \\
 & = (k+1)(3k+1) \left[(n+m-1)^2 \left(\frac{n(n-1)}{2} - m \right) - 2(n+m-1)\overline{M}_1(G) + 4\overline{M}_2(G) \right] \\
 & \quad + (k+1)(3k+1) [(n+m-1)m(n-2) - 2(n-1)2m + 2M_1(G)] \\
 & = (k+1)(3k+1) \left[(n+m-1)^2 \left(\frac{n(n-1)}{2} - m \right) - 2(n+m-1)\overline{M}_1(G) + 4\overline{M}_2(G) \right. \\
 & \quad \left. + (n+m-1)m(n-2) - 2(n-1)2m + 2M_1(G) \right]
 \end{aligned}$$

Corollary 6.5 Let G be a (n, m) -graph and $spl_k(G--)$ be k -splitting graph of $G--$. Then

$$M_2(\overline{spl_k(G--)}) = \frac{1}{2}n'(n'-1)^3 - 3m'(n'-1)^2 + 2m'^2 + \frac{2n'-3}{2}M_1(spl_k(G--)) - M_2(spl_k(G--))$$

where $n' = (n+m)$ and $m' = \frac{1}{2}[n(n-1) + 2m(n-3)](2k+1)$.

Proof. Result follows from Lemma 2.4.

Corollary 6.6 Let G be a (n, m) -graph and $spl_k(G--)$ be k -splitting graph of $G--$. Then

$$\overline{M}_2(spl_k(G--)) = 2m'^2 - \frac{1}{2}M_1(spl_k(G--)) - M_2(spl_k(G--))$$

where $n' = (n+m)$ and $m' = \frac{1}{2}[n(n-1) + 2m(n-3)](2k+1)$.

Proof. Result follows from Lemma 2.5.

Corollary 6.7 Let G be a (n, m) -graph and $spl_k(G--)$ be k -splitting graph of $G--$. Then

$$\overline{M}_2(\overline{spl_k(G--)}) = m'(n'-1)^2 - (n'-1)M_1(spl_k(G--)) + M_2(spl_k(G--))$$

where $n' = (n+m)$ and $m' = \frac{1}{2}[n(n-1) + 2m(n-3)](2k+1)$.

Proof. Result follows from Lemma 2.6.

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