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On Hamilton Laceability and Random Hamiltonian-t^*- Laceablity of Total Transformation Graph G^{-++}

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Abstract: *A connected graph $G = (V, E)$ is called Hamiltonian if G contains a spanning cycle and if a graph G contains a spanning path between arbitrary pair of its vertices is called Hamilton-connected. A bipartite graph is called Hamilton-laceable if there exist Hamiltonian path between vertices of different partite sets and a graph G is random Hamiltonian-t^*- laceable if there exists a $u - v$ Hamiltonian path for at least one pair $u, v \in V(G)$ for t^* distance. In this paper, we have studied the Hamiltonian laceble and random Hamiltonian-t^*- laceable graphs of total transformation graph G^{-++} of graphs viz. path P_n, cycle C_n, complete bipartite graph $K_{r,s}$, n-dimensional convex polytopes D_n, H_n and G_n.*

Keywords: *Hamiltonian graph, Hamiltonian connected, Hamilton-laceable, total transformation graph.*

1. Introduction

Let G be a finite, simple, connected and undirected graph. A cycle in a graph G is called Hamiltonian if it passes through every vertex of G exactly once. Likewise, a Hamiltonian path is a path that visits all the vertices without repetition. However, not all graphs include Hamiltonian cycles. For example, trees are acyclic and therefore cannot contain any Hamiltonian cycles, although they may still possess Hamiltonian paths.

A graph G is classified as Hamiltonian if it has at least one Hamiltonian cycle. By nature, both cycle graphs and complete graphs (also known as cliques) are Hamiltonian. A graph is said to be traceable if it includes at least one Hamiltonian path. While every Hamiltonian graph is necessarily traceable, the reverse is not always true. A classic instance of a traceable but non-Hamiltonian graph is the Petersen graph.

If a graph contains a Hamiltonian path between every pair of distinct vertices, it is referred to as Hamilton-connected. This notion was introduced by Ore[1] in 1963. Frucht [2] investigated trivalent Hamiltonian graphs and their canonical representations.

Bipartite graphs cannot exhibit Hamilton-connectivity, as a Hamiltonian path cannot exist between two vertices in the same partite set. In such cases, if Hamiltonian paths do exist between vertices of opposite partite sets, the graph is termed Hamilton-laceable.

There is a significant amount of research dedicated to the study of Hamiltonicity and Hamilton-connectivity in graphs. Chartrand et al. [8] demonstrated that squaring a block graph results in a Hamilton-connected graph. Thomassen [9] explored Hamilton-connectivity in tournament graphs. Chang et al. [10] analyzed various aspects such as panconnectivity, Hamiltonian fault-tolerance, and Hamilton-connectivity in alternating group graphs, treating them as interconnection network models.

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Kewen et al. [11] identified a sufficient condition for a graph to be Hamilton-connected. Zhou and Wang [12] presented Hamilton-connectivity criteria based on factors like edge count, spectral radius, and signless Laplacian spectral radius.

Further contributions by Zhou et al. [13] involved the computation of the Wiener and Harary indices for Hamilton-connected graphs with considerable diameter. Wei et al. [14] developed spectral versions of Erdős' theorems in the context of Hamilton-connected graphs. Hung et al. [15] delved into Hamilton-connectivity in alphabet grid graphs.

Zhou et al. [16] built upon a result by Fiedler and Nikiforov by proposing signless Laplacian spectral criteria for Hamilton-connectivity in graphs with a large minimum degree. More recently, Shabbir et al. [17] focused on the Hamilton-connectivity properties of Toeplitz graphs.

A finite connected bipartite graph G is called Hamilton-laceable if for any two vertices $v, w \in V(G)$ from different bipartition classes of G there exists a Hamilton-path whose endvertices are v and w.

Laceability in the brick products of even cycles was explored by Alspach et.al. in [22]. A characterization for a 1-connected graph to be Hamiltonian-t-laceable for t = 1; 2 and 3 is given in [10] and this was extended to t = 4 and 5 by Thimmaraju and Murali [34]. Leena Shenoy [33] studied Hamiltonian laceability properties in product graphs involving cycles and paths. More results in the laceability properties of product graphs canbefoundin[27],[28],[29],and[30].

By preserving the vertex-edge incidence relation in convex polytopes, their graphs are constructed. Baca [18] was the first researcher to consider these families of geometric graphs. In [19] Baca studied the problem of magic (resp. graceful and antigraceful) labeling of convex polytopes, whereas, in [20] the problem of face antimagic labeling of convex polytopes was studied. Miller et al. [21] studied the vertex-magic total labeling of convex polytopes.

Definition 1. *A graph $G = (V, E)$ with $|V| \geq 2$, is random Hamiltonian-t^*- laceable if there exists a $u - v$ Hamiltonian path for at least one pair $u, v \in V(G)$ for t^* distance.*

Definition 2. *For $U \subseteq V(G)$ and $v_i, v_j \in V(G)$, if $U = \{u_i; 1 \leq i \leq p\}$ then $v_i \circ \{u_i; 1 \leq i \leq p\} \circ v_j$ means that $v_i \sim u_1$ and $u_p \sim v_j$ and adjacency in the rest of $u_i^{\prime}s$ $(2 \leq i \leq p)$ stays the same.*

Transformation graphs takes information from the original graph and converts source information into a new structure. If it is possible to figure out the given graph from the transformed graph in polynomial time, such operation may be used to survey miscellaneous structural properties of the original graph considering the transformation graphs. Therefore it fosters to study the research of transformation graphs and their structural properties .

Definition 3. *Let $G = (V, E)$ be a graph and x, y, z be three variables taking values $+$ or $-$. The total transformation graph G^{xyz} is a graph having $V(G) \cup E(G)$ as a vertex set, and for $\alpha, \beta \in V(G) \cup E(G)$, α and β are adjacent in G^{xyz} if and only if*

1. $\alpha, \beta \in V(G)$, α, β *are adjacent in* G *if* $x = +$*and* α *and* β *are not adjacent in* G *if* $x = -$
2. $\alpha, \beta \in V(G)$, α, β *are adjacent in* G *if* $y = +$*and* α *and* β *are not adjacent in* G *if* $y = -$
3. $\alpha, \beta \in V(G)$, α, β *are adjacent in* G *if* $z = +$*and* α *and* β *are not adjacent in* G *if* $z = -$

Note 1. Since there are eight distinct 3-permutations of $\{+, -\}$, we obtain eight graphical transformations of G. It is interesting to see that G^{+++} is exactly the total graph $T(G)$ of G and G^{---} is the complement of $T(G)$. Also for a given graph G, G^{++-} and G^{--+}, G^{+-+} and G^{-+-}, G^{-++} and G^{+--} are the other three pairs of complementary graphs.

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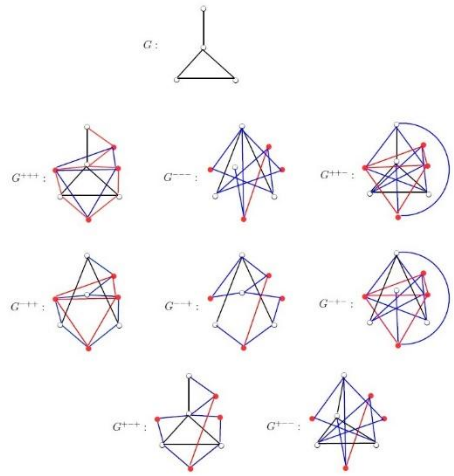

Figure 1. A graph G and its total transformation graphs.

2. Hamiltonicity of G^{-++}

Theorem 1[3]. *G^{-++} is connected for any graph G.*

Theorem 2[3]. *Let G be a graph then $diam(G^{-++}) \leq 3$ and the equality holds if and only if $diam(L(G)) > 2$.*

Theorem 3. *Let G be a graph with of order at least three then G^{-++} is Hamiltonian.*

Proof. Let $G = (V, E)$ be a graph of order n and size m where $n \geq 3$. Clearly, by Theorem A, G^{-++} is connected. Next, we need to show that G^{-++} contains a Hamiltonian path if $|V(G)| \geq 3$. On the contrary, assume that G^{-++} is Hamiltonian and $|V(G)| < 3$. Then either $G = K_1$, K_2, or $\overline{K_2}$, in all these cases, it is easy to check that G^{-++} is not Hamiltonian. Next assume that $|V(G)| \geq 3$ now need to show that G^{-++} is Hamiltonian. Let X be the set of maximum edge-independent set of G, and let H be a graph of order n such that every pair of vertices are adjacent. Clearly, G is a subgraph of H with $|V(G)| = |V(H)|$, and there exists a spanning cycle C in H containing every edge of X. Suppose $e_1, e_2, \ldots, e_n = X'$ are the edges of G on C, then clearly $X \subseteq X'$. Note that if $X' = E(G)$, then it is easy to obtain a spanning cycle of G^{-++} by removing each edge $u_i v_j$ of C by the path $u_i e_i v_j$ of length 2 for $i = 1,2, \ldots, m$. Otherwise, $E(G) \setminus X'$ is non-empty, since X is the maximum set of independent edges of G, then $E(G) \setminus X'$ is incident to at least some vertex, say v. To get the spanning cycle for G^{-++}, we should include all the edges of G into C. By replacing every edge of C by a path, we get a Hamiltonian cycle for G^{-++}. This completes the proof.

Theorem 4. Let $G = P_n; n \geq 3$, then

$$P_n^{-++} \begin{cases} random\ Hamiltonian\ 2^* - laceable, & \text{if}\, n \leq 3,4; \\ random\ Hamiltonian\ 3^* - laceable, & n \geq 5. \end{cases}$$

Proof. Let $G = P_n$ be a path with $n \geq 3$. Let $V(P_n) = \{v_1, v_2, v_3, \cdots, v_n\}$ and $E(P_n) = \{e_1, e_2, e_3, \cdots, e_m\}$. Clearly by Theorem 1, P_n^{-++} is connected and by Theorem 2, $diam(P_n^{-++}) \leq 3$. Also from Theorem 3, P_n^{-++} Hamiltonian. Hence P_n^{-++} is either random Hamiltonian 2^*- laceable or random Hamiltonian 3^*-laceable. Consider the following cases:

$$P_3^{-++}: v_1 \sim v_3 \sim e_2 \sim e_1 \sim v_2$$

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Hence, P_3^{-++} is random Hamiltonian 2^*- laceable.

Let $n = 4$, then $G = P_4$ with $V(P_4) = \{v_1, v_2, v_3, v_4\}$ and $E(P_n) = \{e_1, e_2, e_3\}$. Then $V(P_4^{-++}) = \{v_1, v_2, v_3, v_4, e_1, e_2, e_3\}$. Clearly, $d(v_1, v_2) = 2$. We need to show that P_4^{-++} contains $v_1 - v_2$ Hamiltonian path. i.e.

$$P_4^{-++}: v_1 \sim v_4 \sim e_3 \sim v_3 \sim e_2 \sim e_1 \sim v_2$$

Hence, P_4^{-++} is random Hamiltonian 2^*- laceable.

$$\mathrm{P}_{\mathrm{n}}^{\{-++\}}: \mathrm{e}_1 \sim \mathrm{v}_2 \sim \mathrm{e}_2 \sim \mathrm{v}_3 \sim \mathrm{e}_3 \sim \cdots \sim \mathrm{e}_{\mathrm{m}-1} \sim \mathrm{v}_{\mathrm{n}-1} \sim \mathrm{v}_1 \sim \mathrm{v}_{\mathrm{n}} \sim \mathrm{e}_m$$

Hence, P_n^{-++} is random Hamiltonian 3^*- laceable.

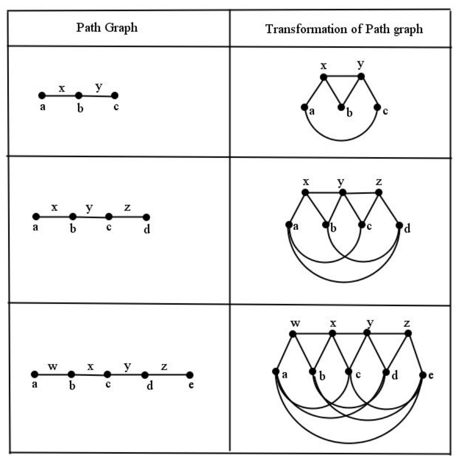

Figure 2. A graph G and its total transformation graphs.

Theorem 5. Let $G = C_n; n \geq 3$, then

$$C_n^{-++} \begin{cases} random\ Hamiltonian\ 2^* - laceable, & \text{if } n \leq 3,4,5; \\ random\ Hamiltonian\ 3^* - laceable, & n \geq 6. \end{cases}$$

Proof. Let $G = C_n$ be a cycle with $n \geq 3$. Let $V(C_n) = \{v_1, v_2, v_3, \cdots, v_n\}$ and $E(C_n) = \{e_1, e_2, e_3, \cdots, e_m\}$. Clearly by Theorem 1, C_n^{-++} is connected and by Theorem 2, $diam(C_n^{-++}) \leq 3$. Also from Theorem 3, C_n^{-++} Hamiltonian. Hence C_n^{-++} is either random Hamiltonian 2^*- laceable or random Hamiltonian 3^*- laceable. Consider the following cases:

$$C_3^{-++}: v_1 \sim e_3 \sim v_3 \sim e_3 \sim e_1 \sim v_2$$

Hence, C_3^{-++} is random Hamiltonian 2^*- laceable.

$$C_4^{-++}: v_1 \sim e_1 \sim e_4 \sim v_4 \sim e_3 \sim v_3 \sim e_2 \sim v_2$$

Hence, C_4^{-++} is random Hamiltonian 2^*- laceable.

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$$C_5^{-++}: v_1 \sim e_1 \sim e_5 \sim v_5 \sim e_4 \sim v_4 \sim e_3 \sim v_3 \sim e_2 \sim v_2$$

Hence, C_5^{-++} is random Hamiltonian 2^*- laceable.

$$C_n^{-++}: e_1 \sim v_1 \sim e_m \sim v_n \sim e_{m-1} \sim v_{n-1} \sim e_{m-2} \sim v_{n-2} \sim \cdots \sim v_3 \sim e_3 \sim e_2 \sim v_2 \sim v_4 \sim e_4$$

Hence, C_n^{-++} is random Hamiltonian 3^*- laceable.

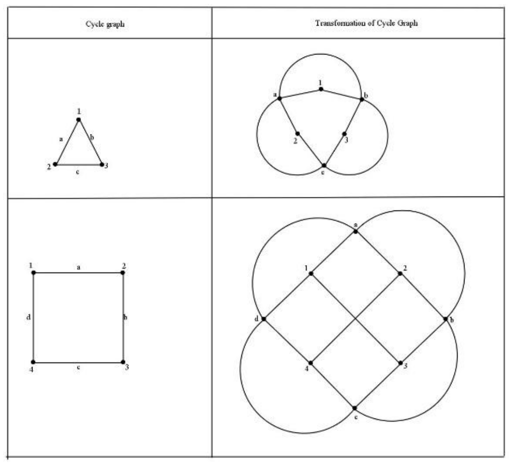

Figure 3. A graph G and its total transformation graphs.

Theorem 6. Let $G = K_{r,s}; 2 \le r \le s$, then $K_{r,s}^{-++}$ is Hamiltonian connected.

Proof. Let $G = K_{r,s}; 2 \le r \le s$ with $V(K_{r,s}) = \{v_1, v_2, v_3, \cdots, v_r, u_1, u_2, \cdots, u_s\}$, $E(K_{r,s}) = \{e_1, e_2, e_3, \cdots, e_{rs}\}$ and $H = K_{r,s}^{-++}$. We prove this result by definition. For this, we have to show that there exist Hamiltonian paths between any pair of vertices of H. Let $P_H(u, v)$ be a Hamiltonian path between vertices u and v in H. Let $V(H) = V_r \cup V_s \cup E_{rs}$, where $V_r = \{v_1, v_2, v_3, \cdots, v_r\}$, $V_s = \{u_1, u_2, \cdots, u_s\}$ and $E_{rs} = \{e_1, e_2, e_3, \cdots, e_{rs}\}$.

The existence of the Hamiltonian path between every pair of vertices of the H completes the proof.

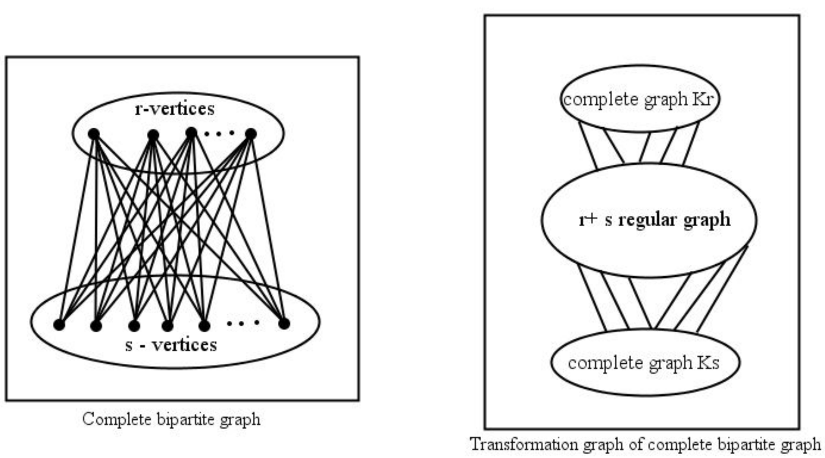

Figure 4. A graph G and its total transformation graphs.

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Theorem 7.Let $G = K_{1,n-1}; n \geq 4$, then $K_{1,n-1}^{-++}$ is Hamiltonian connected.

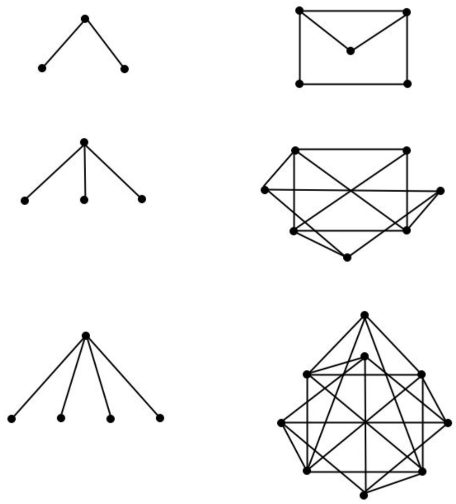

Figure 5. A graph G and its total transformation graphs.

Theorem 8. Let D_n denote the n-dimensional convex polytope with $n \geq 4$, then $D_n^{-++}$is random Hamiltonian t^*-laceable.

Proof. The vertex set of D_n consists of four layers of vertices, i.e., w_p, x_p, y_p, and z_p. That is to say that $V(D_n) = \{w_p, x_p, y_p, z_p: 1 \leq p \leq n\}$. Accordingly, the edge set of D_n is as follows:

$$E(D_n) = \{w_p w_{p+1}, z_p z_{p+1}, w_p x_p, x_p y_p, x_{p+1} y_p, y_p z_p: 1 \leq p \leq n\}. \quad (3)$$

The subscripts are to be considered modulo n. The layer of vertices comprising w_p is called the inner layer, whereas the layer comprising z_p is called the outer layer of D_n. The vertices x_p and $y_p, 1 \leq p \leq n$ form the middle layers. Let D_n^{-++} denote the generalized transformation graph of D_n. By the definition, The vertex set and the edge set of D_n^{-++} is given by:

$$V(D_n^{-++}) = \{w_p, x_p, y_p, z_p: 1 \leq p \leq n\} \cup \{w_p w_{p+1}, z_p z_{p+1}, w_p x_p, x_p y_p, x_{p+1} y_p, y_p z_p: 1 \leq p \leq n\}$$

and $|E(D_n^{-++})| = |E_1| + |E_2| + |E_3|$, where

$$\begin{aligned} E_1(D_n^{-++}) &= \{z_i z_j\} \cup \{z_i, y_j\} \cup \{z_i, x_j\} \cup \{z_i, w_j\} \cup \{y_i, y_j\} \cup \{y_i, x_j\} \cup \{y_i, w_j\} \\ &\cup \{x_i, x_j\} \cup \{x_i, w_j\} \cup \{w_i, w_j\} \end{aligned}$$

where, $z_i \nsim z_j$, $z_i \nsim y_j$, $z_i \nsim x_j$, $z_i \nsim w_j$, $y_i \nsim y_j$, $y_i \nsim x_j$, $y_i \nsim w_j$, $x_i \nsim x_j$, $x_i \nsim w_j$ and $w_i \nsim w_j$.

$$\begin{aligned} E_2(D_n^{-++}) &= -|\{w_p w_{p+1}, z_p z_{p+1}, w_p x_p, x_p y_p, x_{p+1} y_p, y_p z_p: 1 \leq p \leq n\}| \\ &+ \frac{1}{2}\left(\sum_{i=1}^{|V(D_n)|} deg_{D_n}(v_i)^2\right) \end{aligned}$$

$$E_3(D_n) = 2|\{w_p w_{p+1}, z_p z_{p+1}, w_p x_p, x_p y_p, x_{p+1} y_p, y_p z_p: 1 \leq p \leq n\}|$$

Now, by Theorem 2, $diam(D_n^{-++}) \leq 3$. Hence we need to show that there exists a pair of vertices $v_i, v_j \in D_n^{-++}$ such that $d(v_i, v_j) = 3$ such that $P_H(v_i, v_j)$ exists in D_n^{-++}. By looking at the structure of

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D_n it is obvious that the outer layer $\bigcup_{i=1}^{n} z_i$ will form a cycle in D_n^{-++}, therefore, $u = z_i z_j$ and $v = z_k z_l$ are the vertices at a maximum distance 3 in D_n^{-++}. Hence by Proposition 5, the result follows.

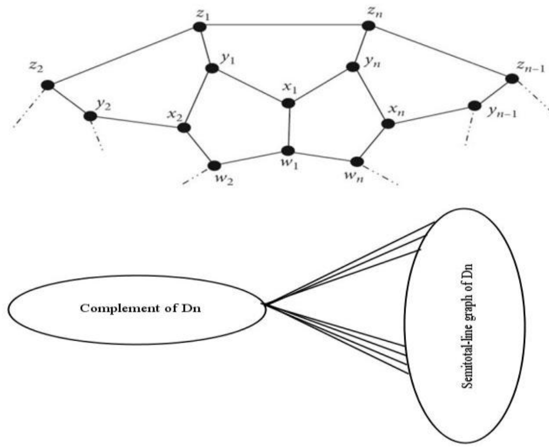

Figure 6. A graph G and its total transformation graphs.

Theorem 9. Let H_n denote the n-dimensional convex polytope with $n \geq 5$, then $H_n^{-++}$is random Hamiltonian t^*- laceable.

Proof. the vertex set of H_n consists of four layers of vertices, i.e., v_p, w_p, x_p, y_p, and z_p. That is to say that $V(H_n) = \{v_p, w_p, x_p, y_p, z_p: 1 \leq p \leq n\}$. Accordingly, the edge set of H_n is as follows:

$$E(H_n) = \{v_p v_{p+1}, v_p w_p, w_p v_{p+1}, w_p w_{p+1}, w_p x_p, x_p x_{p+1}$$

The subscripts are to be considered modulo n. See Figure 2 to view the n-dimensional convex polytope graph H_n. Let H_n^{-++} denote the generalized transformation graph of H_n. By the definition, The vertex set and the edge set of H_n^{-++} is given by:

$$\begin{aligned} V(H_n^{-++}) \quad &= \quad \{v_p, w_p, x_p, y_p, z_p: 1 \leq p \leq v\} \cup \{v_p v_{p+1}, v_p w_p, w_p v_{p+1}, w_p w_{p+1}, w_p x_p, x_p x_{p+1}\} \\ &\quad\quad \{x_p y_p, y_p x_{p+1}, y_p y_{p+1}, y_p z_p, z_p z_{p+1}: 1 \leq p \leq n\} \end{aligned}$$

and $|E(H_n^{-++})| = |E_1| + |E_2| + |E_3|$, where

$$\begin{aligned} E_1(H_n^{-++}) \quad &= \quad \{z_i z_j\} \cup \{z_i y_j\} \cup \{z_i x_j\} \cup \{z_i w_j\} \cup \{z_i v_j\} \\ &\cup \quad \{y_i y_j\} \cup \{y_i x_j\} \cup \{y_i w_j\} \cup \{y_i v_j\} \\ &\cup \quad \{x_i x_j\} \cup \{x_i w_j\} \cup \{x_i v_j\} \\ &\cup \quad \{w_i w_j\} \cup \{w_i v_j\} \cup \{v_i v_j\} \end{aligned}$$

where, $z_i \nsim z_j$, $z_i \nsim y_j$, $z_i \nsim x_j$, $z_i \nsim w_j$, $z_i \nsim v_j$, $y_i \nsim y_j$, $y_i \nsim x_j$, $y_i \nsim w_j$, $y_i \nsim v_j$, $x_i \nsim x_j$, $x_i \nsim w_j$, $x_i \nsim v_j$, $w_i \nsim w_j$, $w_i \nsim v_j$ and $v_i \nsim v_j$.

$$E_2(H_n^{-++}) \quad = \quad -|E(H_n)| + \frac{1}{2}\left(\sum_{i=1}^{|V(H_n)|} deg_{H_n}(v_i)^2\right)$$

$$E_3(H_n) \quad = \quad 2|E(H_n)|$$

Now, by Theorem 2, $diam(H_n^{-++}) \leq 3$. Hence we need to show that there exists a pair of vertices $v_i, v_j \in H_n^{-++}$ such that $d(v_i, v_j) = 3$ such that $P_H(v_i, v_j)$ exists in H_n^{-++}. By looking at the structure of

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H_n it is obvious that the outer layer $\bigcup_{i=1}^{n} z_i$ will form a cycle in H_n^{-++}, therefore, $u = z_i z_j$ and $v = z_k z_l$ are the vertices at a maximum distance 3 in H_n^{-++}. Hence by Proposition 5, the result follows.

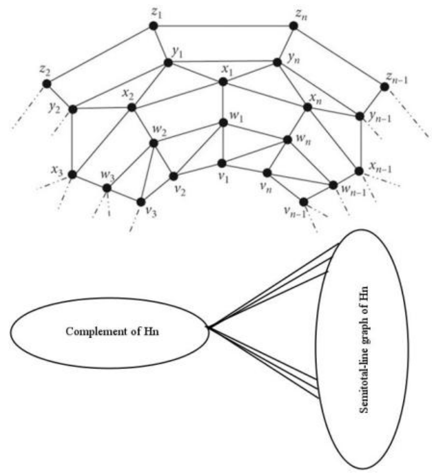

Figure 7. A graph G and its total transformation graphs.

Theorem 10. Let G_n denote the n-dimensional convex polytope with $n \geq 5$, then $G_n^{-++}$is random Hamiltonian t^*- laceable.

Proof. The vertex set of G_n consists of four layers of vertices, i.e., w_p, x_p, y_p, and z_p. That is to say that $V(G_n) = \{w_p, x_p, y_p, z_p: 1 \leq p \leq n\}$. Accordingly, the edge set of G_n is as follows:

$$E(G_n) = \{w_p w_{p+1}, x_p x_{p+1}, y_p y_{p+1}, z_p z_{p+1}, w_p x_p, x_p y_p,$$

The subscripts are to be considered modulo v. Figure 3 presents the n-dimensional convex polytope G_n with proper labeling of vertices which will be used to show its Hamilton connectivity. Let G_n^{-++} denote the generalized transformation graph of G_n. By the definition, The vertex set and the edge set of G_n^{-++} is given by:

$$\begin{aligned} V(G_n^{-++}) &= \{w_p, x_p, y_p, z_p: 1 \leq p \leq n\} \cup \{w_p w_{p+1}, x_p x_{p+1}, y_p y_{p+1}, z_p z_{p+1}, w_p x_p, x_p y_p, \}. \\ & \{. y_p x_{p+1}, y_p z_p: 1 \leq p \leq n-1\} \end{aligned}$$

and $|E(G_n^{-++})| = |E_1| + |E_2| + |E_3|$, where

$$\begin{aligned} E_1(G_n^{-++}) &= \{z_i z_j\} \cup \{z_i, y_j\} \cup \{z_i, x_j\} \cup \{z_i, w_j\} \cup \{y_i, y_j\} \cup \{y_i, x_j\} \cup \{y_i, w_j\} \\ \cup & \quad \{x_i, x_j\} \cup \{x_i, w_j\} \cup \{w_i, w_j\} \end{aligned}$$

where, $z_i \nsim z_j$, $z_i \nsim y_j$, $z_i \nsim x_j$, $z_i \nsim w_j$, $y_i \nsim y_j$, $y_i \nsim x_j$, $y_i \nsim w_j$, $x_i \nsim x_j$, $x_i \nsim w_j$ and $w_i \nsim w_j$.

$$\begin{aligned} E_2(G_n^{-++}) &= -|E(G_n)| + \frac{1}{2}\Big(\sum_{i=1}^{|V(G_n)|} deg_{G_n}(v_i)^2\Big) \\ E_3(G_n) &= 2|E(G_n)|. \end{aligned}$$

Now, by Theorem 2, $diam(G_n^{-++}) \leq 3$. Hence we need to show that there exists a pair of vertices $v_i, v_j \in G_n^{-++}$ such that $d(v_i, v_j) = 3$ such that $P_H(v_i, v_j)$ exists in G_n^{-++}. By looking at the structure of

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G_n it is obvious that the outer layer $\bigcup_{i=1}^{n} z_i$ will form a cycle in G_n^{-++}, therefore, $u = z_i z_j$ and $v = z_k z_l$ are the vertices at a maximum distance 3 in G_n^{-++}. Hence by Proposition 5, the result follows.

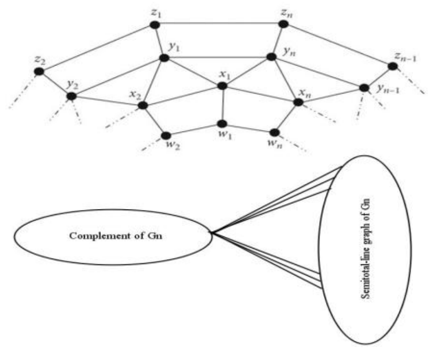

Figure 8. A graph G and its total transformation graphs.

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