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Abstract: *The distance between two nodes x_1, x_2 contained in V(G) is the minimum size of $x_1 – x_2$ paths in G. An $x_1 – x_2$ path of the size $d_G(x_1, x_2)$ is known as geodesic. We indicate $I_G(x_1, x_2)$ as the set of nodes which are lies inside some $x_1 – x_2$ geodesics of G. A node is said to lie on $x_1 – x_2$ geodetic if c is an inner node of P. The bounded interval $I(x_1, x_2)$ includes x_1, x_2 and all nodes lying on some $x_1 – x_2$ geodesic of G. Consider a non- empty set $S \subseteq V(G)$. For a set $I(S) = \bigcup_{x_1, x_2 \in S} I(x_1, x_2)$. If G is connected graph, thus S is a geodetic set with the condition that I(S) = V (G) [2]. The minimum cardinality S of G is called geodetic number defined by g(G). A j – node coloring of G is an allotment of j colors to the nodes of G. The coloring is proper if no two joining nodes accept same color such that $\chi(G) = j$ is said to be j – chromatic, where $j \leq k$. a minimum cardinality of a chromatic number of G is called chromatic set [1, 3, 4]. Beulah Samli and Robinson Chellathurai [9] introduced the concepts of the Geo Chromatic number of graph. In this paper, we consider a graph to be connected, finite, simple where V(G) is node set and E(G) is link set [5,6]. In this paper, we have defined geo chromatic number for certain graph and have discussed few results.*

Keywords: *Geodetic Number, Geo chromatic number, Lexicographic product, Tensor Product, Co-Normal Product*

1. Introduction

In this paper, we find the geo chromatic number for Lexicographic product graph or Composition graph, Tensor product graph and Co-Normal product graph.

1. 2. Geo chromatic number of Lexicographic Product or Composition of Graphs

Theorem1.2.1. For the Lexicographic product of two path graphs P_n and P_m, where m and n are integers m > 2 and n = 2, the geo chromatic number is given by,

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$$\chi_{gc}(P_n \bullet P_m) = \begin{cases} \left\lceil \frac{m}{2} \right\rceil + 3, \text{if } m \text{ is odd} \\ \left\lceil \frac{m+1}{2} \right\rceil + 2, \text{if } m \text{ is even} \end{cases}.$$

Proof: Let the vertices of $P_n(n=2)$ be $\{u_1,u_2\}$and the vertices of P_m be$\{v_1, v_2, v_3, \ldots, v_m\}$.Let us denote $G = P_n \bullet P_m$ as the lexicographic product of path graphs P_n and P_m. The graph $P_n \bullet P_m$ is 4 - colorable, Hence $\chi_{gc}(P_n \bullet P_m) = 4$.

Case(i): When m is odd, the geodetic set S is given by $S=\{(u_1,v_1), (u_1,v_3), (u_1,v_5), \ldots, (u_1,v_m)\}$which is minimum. $|S| = \left\lceil \frac{m}{2} \right\rceil \neq \chi_{gc}(P_n \bullet P_m)$as the set S receives only one color. By adding 3 vertices of different colors to the set S, it becomes a geo chromatic set. That is $S_c = S \cup$ {three vertices with distinct colors} will be a minimum geo chromatic set. Hence it results $\chi_{gc}(P_n \bullet P_m) = \left\lceil \frac{m}{2} \right\rceil + 3$. .

Case(ii): When m is even, the geodetic set S is given by $S=\{(u_1,v_1), (u_1, v_3), (u_1,v_5), \ldots, (u_1,v_{m-1}), (u_1,v_m)\}$which is minimum and $|S| = \left\lceil \frac{m+1}{2} \right\rceil$.The graph $P_n \bullet P_m$ is 4-colorable. But the set S receives two different colors c_1 and c_2 (say) and not making S a chromatic set. To make S, a chromatic set, we add two vertices with colors c_3 and c_4 to the set S, thus S becomes a minimum geo chromatic set, thereby it results $\chi_{gc}(P_n \bullet P_m) = \left\lceil \frac{m+1}{2} \right\rceil$.

Corollary 1.2.2. For the Lexicographic product of two path graphs P_3 and P_3, the geo chromatic number is given by $\chi_{gc}(P_n \bullet P_m) = 6$.

Proof: We label the vertices of graph $G = P_3$ by $V(G) = \{u_1, u_2, u_3\}$ and the vertices of $H = P_3$ by $V(H) = \{v_1,v_2,v_3\}$. It is clear that the lexicographic product has 32 vertices with $\triangle[P_3 \bullet P_3] = 8$ and $\delta[P_3 \bullet P_3] = 4$, the corner vertices are exactly of degree 4, the remaining vertices are of degree 5 and 7 and the centre vertex is of degree 8. Now, for the lexicographic product, $V(P_3 \bullet P_3) = \{(u_1,v_1), (u_1,v_2), (u_1,v_3), (u_2,v_1), (u_2,v_2), (u_2,v_3), (u_3,v_1), (u_3,v_2), (u_3,v_3)\}$ will be the vertex set. The set $S=\{(u_1,v_1), (u_1,v_3), (u_3,v_1), (u_3,v_3)\}$ is a minimum geodetic set and $|S| = 4$. By proper coloring, vertices $\{(u_1,v_1), (u_1,v_3), (u_3,v_2)\}$ may receive color c_1, $\{(u_2, v_1), (u_2, v_3)\}$ may receive color c_2, $\{(u_1, v_2),$

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(u_3,v_1), $(u_3,v_3)\}$ may receive color c_3 and the center vertex $\{(u_2,v_2)\}$ may receive color c_4. But the set S receive the same color sayc_1, thus it is not a geo chromatic set. Hence, we add two vertices with the remaining colors c_2 and c_3 to the set S. Therefore $\chi_{gc}(P_n \bullet P_m) = 6$.

Corollary 1.2.3. For the Lexicographic product of two path graphs, the geo chromatic number of $P_n \bullet P_m$ is 6 when m = n=3.

Remark 1.2.4. The Lexicographic product of graphs or Composition of graphs is in general non commutative $G \bullet H \neq H \bullet G$

Theorem 1.2.5. For the Lexicographic product of two path graphs $P_n \bullet P_m$ where m = n and m, n > 3, the geo chromatic number is given by

$$\chi_{gc}(P_n \bullet P_m) = \begin{cases} 4, \textit{ if n and m are even} \\ 7, \textit{ if n and m are odd}. \end{cases}$$

Proof: Consider the vertices of P_n by $\{u_1, u_2, u_3, ..., u_n\}$ and the vertices of P_m by$\{v_1, v_2, v_3, ...,v_m\}$. Let us denote the lexicographic product of two path graphs as $G = P_n \bullet P_m$, which is 4 - colorable. The geodetic set of graph G is $S = \{(u_1,v_1), (u_1,v_m), (u_n,v_1), (u_n,v_m)\}$ which is minimum. That is $|S|= 4$.

Case (i): When m and n are even, S is a minimum geodetic set consists of 4 different colors, also making it a chromatic set. Hence $\chi_{gc}(P_n \bullet P_m) = 4$.

Case (ii): When m and n are odd, $S = \{(u_1,v_1), (u_1,v_m), (u_n,v_1), (u_n,v_m)\}$ is a minimum geodetic set whose elements have the same color say c_1, hence S is not a chromatic set. To make the set S chromatic, we add three vertices, which are neighbors to the set S with three colors other than c_1.The vertices $\{(u_1,v_2), (u_2,v_1), (u_1,v_{m-1})\}$which are neighbors to the set S are added to make S a chromatic set. Thus $S_c = S \cup \{(u_1,v_2), (u_2,v_1), (u_1,v_{m-1})\}$is both minimum geodetic and chromatic set. Hence $\chi_{gc}(P_n \bullet P_m) = 7$.

Theorem 1.2.6. For the Lexicographic product of Cycle graph C_n with Path graph P_2, the geo chromatic number is given by $\chi_{gc}(C_n \bullet P_2) = 6$.

Proof. Consider the vertices of C_n by $\{u_1, u_2, u_3, ..., u_n\}$ and the vertices of P_2 by $\{v_1,$

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v_2}. The cycle graph C_n is 2 – colourable if n even and it is 3–colourable if n is odd. The lexicographic product, G = $C_n \bullet P_2$ has $\chi(G)$ = 6. The set

$$S=\left\{(u_1,v_1),\left(u_{\frac{n+1}{2}},v_1\right),(u_n,v_1),(u_1,v_2),\left(u_{\frac{n+1}{2}},v_2\right),(u_n,v_2)\right\} \text{ or}$$

$$S=\left\{(u_1,v_1),\left(u_{\frac{n}{2}+1},v_1\right),(u_1,v_2),\left(u_{\frac{n}{2}+1},v_2\right)\right\}$$ will be a minimum geodetic set of graph G.

Case(i): when n is odd, the set $S=\left\{(u_1,v_1),\left(u_{\frac{n+1}{2},}v_1\right),(u_n,v_1),(u_1,v_2),\left(u_{\frac{n+1}{2},}v_2\right),(u_n,v_2)\right\}$ contains vertices with 6 different colors, therefore the set S is a minimum geo chromatic set of graph G and |S|=6. Hence $\chi_{gc}(C_n \bullet P_2)$= 6.

Case (ii): when n is even, the set $S=\left\{(u_1,v_1),\left(u_{\frac{n}{2}+1},v_1\right),(u_1,v_2),\left(u_{\frac{n}{2}+1},v_2\right)\right\}$ is a minimum geodetic set and receives only four different colors and |S|=4. That is |S| < |χ($C_n \bullet P_2$)|. By adding the neighborhood vertices (u_1, v_1) and (u_1, v_2) to the set S, S_c= S∪ {(u_1,v_1), (u_1,v_2)} is a minimum geo chromatic set of the graph. Thus,
$\chi_{gc}(C_n \bullet P_2)$ = 6.

1.3. Geo Chromatic Number of Tensor Product of Graphs

Remark 1.3.1. The distance between two vertices u and v in a connected graph G is the length of the shortest u–v path in graph G, the u–v length is called u–v geodesic. The tensor product of two paths P_n and P_m is a disconnected graph. Therefore, the geo chromatic number for tensor product of two path graphs does not exist.

Theorem 1.3.2. The tensor product of cycle graph C_m, (m is odd) with path graph P_2, has the geo chromatic number $\chi_{gc}(C_m \otimes P_2)$ =2.

Proof. We label the vertices of C_m by {u_1, u_2, u_3, ..., u_m} and vertices of P_2 by{v_1, v_2}.The cycle graph C_m is 3 – colorable if m is odd and P_2 is 2–colorable. The tensor product graph G = $C_m \otimes P_2$ is an even cycle graph. Thus χ(G) = 2. The geodetic set S = {(u_1,v_1), (u_1,v_2)} is minimum and also a chromatic set. Therefore, it results

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$\chi_{gc}(C_m \otimes P_2)$=2.

Remark 1.3.3. The geo chromatic number of Tensor product of C_m and P_2, does not exist, when m is even.

Corollary 1.3.4. For the Tensor product graph of cycle graph C_m, with path graph P_2, the geo chromatic number $\chi_{gc}(C_m \otimes P_2)= \begin{cases} 2, & \text{if } m \text{ is odd} \\ \text{does not exist}, & \text{if } m \text{ is even} \end{cases}$

Theorem 1.3.5. For the Tensor product of cycle graph C_m, with path graph P_n, the geo chromatic number $\chi_{gc}(C_m \otimes P_2)$= n, for m is odd and n > 1.

Proof. We label the vertices of C_m by {u_1, u_2, u_3, ..., u_m} where m is odd and the vertices of P_n by {v_1, v_2, v_3, ..., v_n}. The cycle graph C_m is 3–colorable since m is odd and P_n is 2 – colorable. The Tensor product graph G = $C_m \otimes P_n$ is 2 colorable. The geodetic set S = $\left\{\left(u_{\frac{m+1}{2}}, v_i\right), 1 \le i \le n\right\}$ is minimum and also a chromatic set. Hence $\chi_{gc}(C_m \otimes P_2)$ = n.

1.4. Geo Chromatic Number of Co-Normal Product Graphs

Theorem 1.4.1. For the Co-Normal product of two path graphs P_m (m>3) and P_n (n=2), the geo chromatic number is given by $\chi_{gc}(P_m * P_n)= \begin{cases} \frac{n+m}{2}+2, & \text{if } m \text{ is even} \\ \frac{n+m-1}{2}+3, & \text{if } m \text{ is odd} \end{cases}$.

Proof. Now consider the vertices of P_m by {u_1, u_2, u_3, ..., u_m}and vertices of P_2 by {v_1, v_2}. Let us denote the co-normal product of these two path graphs as G = $P_m * P_n$. The graph G is 4 – colorable and thus χ(G) = 4. The geodetic set of graph G is S = {(u_i,v_1): I = 1, 3, 5,...., 2m – 1} when m is odd and S = {(u_i,v_1), (u_{m-1},v_1): i = 1, 3, 5,...., 2m – 1} when m is even.

Case (i): When m is even, S = {(u_i,v_1), (u_{m-1},v_1): i = 1, 3 , 5,., 2m – 1} is minimum geodetic but not a chromatic set of graph G, where (u_i,v_1) receives color c_1 and (u_{m-1},v_1) receives color c_2 and |S| = $\frac{n+m}{2}$. Since χ(G) = 4, add two vertices with two different

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colors to the set S. Hence it results, $\chi_{gc}(P_m * P_n) = \frac{n+m}{2}+2$ for n=2.

Case(ii): When m is odd, S = {(u_i,v_1): i = 1, 3, 5, ..., 2m – 1} is minimum geodetic but not a chromatic set and |S| = $\frac{n+m-1}{2}$. The vertex (u_i,v_1) receives color c_1. Since χ(G) = 4, by adding 3 vertices with different colors to the set S, it becomes minimum geo chromatic set. Hence its results $\chi_{gc}(P_m * P_n) = \frac{n+m-1}{2}+3$, for n = 2.

Corollary 1.4.2. For the CoNormal product of cycle graph C_m with path graph P_2, the geo chromatic number of (C_m∗P_2) is 4 when m=3.

Theorem 1.4.3. For the Co-Normal product of C_m, (m >3) and P_2 (n = 2), the geo chromatic number is given by $\chi_{gc}(C_m * P_2) = \begin{cases} \frac{m-1}{2}+1, & \text{if } m \text{ is odd} \\ \frac{m}{2}+3, & \text{if } m \text{ is even.} \end{cases}$.

Proof. Consider the vertices of C_m as {u_1, u_2, u_3,....,u_m} and the vertices of P_2 by{v_1, v_2}. The co-normal product of C_m and P_2 is denoted as C_m∗ P_2. It is known that $\chi(C_m * P_2)$= 4. The geodetic set S is given by S = {(u_i,v_1): i=1, 3, 5,...., 2m – 1}, for m is even and S = {(u_i,v_1), (u_{m-1},v_1): i = 1, 3, 5,....,2m-1}, for m is odd. Then the following two cases arise

Case(i): When m is odd, S = {(u_i,v_1),(u_{m-1},v_1): i = 1, 3, 5,....,2m – 1} is minimum (u_{m-1}, v_1) receive color c_4 and |S| = $\frac{m-1}{2}-1$. Therefore, the set S is not a chromatic set. Hence by adding vertices with the remaining two colors to the set S, it becomes a minimum geo chromatic set. Therefore it results $\chi_{gc}(C_m * P_2) = \frac{m-1}{2}+1$.

Case(ii): When m is even, S = {(u_i,v_1) : i = 1, 3, 5,...., 2m – 1} is minimum geodetic and S receives colourc_1, therefore the set S is not a chromatic set and |S| = $\frac{m}{2}$. Hence by adding the remaining three colors to the set S it becomes a minimum geo chromatic set. Therefore, it results $\chi_{gc}(C_m * P_2) = \frac{m}{2}+3$.

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