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ARITHMETIC OPERATIONS OF GENERALIZED PICTURE FUZZY NUMBERS BY $(\alpha, \gamma, \beta) -$ CUT METHOD

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Abstract: Picture fuzzy set is an extension of intuitionistic fuzzy set and fuzzy set and it is capable to analyze the uncertain and vague information more effectively. It is applicable in situations where human decisions claim more variety of responses like yes, no, abstain and refusal which cannot be addressed in the traditional fuzzy set and intuitionistic fuzzy set. In this article, the arithmetic operations such as scalar multiplication of a generalized trapezoidal and triangular picture fuzzy numbers and the division and multiplication of two generalized trapezoidal and triangular picture fuzzy numbers by $(\alpha, \gamma, \beta) -$ cut method are discussed thoroughly. Finally, some numerical examples are described to illustrate the proposed operations.

Keywords: Picture fuzzy set; Generalized trapezoidal picture fuzzy number; Generalized triangular picture fuzzy number; $(\alpha, \gamma, \beta) -$ cut; Arithmetic operations; $(\alpha, \gamma, \beta) -$ cut method.

1. Introduction

In 1965, Zadeh [30] proposed the notion of fuzzy set as an extension of the crisp set to deal with imprecise and unclear information in ambiguous situations and it is effective and acceptable. Then in 1986, Atanassov [1] introduced intuitionistic fuzzy set which is the generalization of fuzzy set and characterized by a membership function as well as a non-membership function. Although intuitionistic fuzzy sets have vast applications in many fields, it cannot provide all the information in real life situation. For example, in the voting process, we can vote in favor of someone, abstain, against someone and even refuse to cost the vote, which cannot be handled by intuitionistic fuzzy sets. To tackle such type of situations, Cuong [7, 8] gave the idea of picture fuzzy set, which is generalized structure of fuzzy set and intuitionistic fuzzy set and further examined their basic properties and laws. In picture fuzzy set theory, positive, neutral and negative membership degrees of each element belonging to set are considered. Therefore, the picture fuzzy set is an efficient mathematical model to deal with uncertain real life problems, in which an intuitionistic fuzzy set may fail to reveal satisfactory results. The concept of fuzzy numbers was introduced by Chang and Zadeh [4] with some arithmetic operations. The arithmetic operations of fuzzy number are the extensions of the operations of classical interval arithmetic operations which was introduced by R.E. Moore [21]. Then a number of researchers studied the concept of fuzzy numbers with their arithmetic operations [2, 6, 10, 11, 12, 27, 28, 29, 31]. The concept of intuitionistic fuzzy number was proposed by Burillo [3]. Mahapatra and Roy [18] presented triangular intuitionistic fuzzy number and used it for reliability evaluation. In 2010, they [19] also defined trapezoidal intuitionistic fuzzy number and discussed its arithmetic operations based on (α, β) −cut method. Later numerous works have been done on intuitionistic fuzzy number and applied in many branches of science and engineering [16, 17, 20, 22, 23, 24, 25, 26]. Chakraborty et al. [5] discussed arithmetic operation of intuitionistic fuzzy numbers thoroughly. P. Dutta. et al. [9] discussed the operations on picture fuzzy numbers and applied in Multi-Criteria Group Decision Making Problems. Hasan el al. [14, 15] discussed the arithmetic operations such as addition,

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subtraction and scalar multiplication of both generalized triangular and trapezoidal picture fuzzy numbers with some applications.

In this article, the scalar multiplication of a generalized trapezoidal and triangular picture fuzzy number and the division and multiplication of two generalized trapezoidal and triangular picture fuzzy numbers by $(\alpha, \gamma, \beta)-$ cut method are thoroughly discussed with some illustrations.

The article is organized as follows: In section 2, some basic definitions and operations are given which are essential to rest of the paper. In section 3, the scalar multiplication of a generalized trapezoidal and triangular picture fuzzy number and the division and multiplication of two generalized trapezoidal and triangular picture fuzzy numbers by $(\alpha, \gamma, \beta)-$ cut method are discussed. In section 4, numerical examples are described to illustrate the proposed operations. In section 5, the conclusion remark is given.

2. Preliminaries

Definition 2.1 ([30]). Let X be non-empty set. A fuzzy set A in X is given by

$$A = \{(x, \mu_A(x)): x \in X\},$$

where $\mu_A: X \to [0, 1]$.

Definition 2.2 ([1]). An intuitionistic fuzzy set A in X is given by

$$A = \{(x, \mu_A(x), \nu_A(x)): x \in X\},$$

where $\mu_A: X \to [0, 1]$ and $\nu_A: X \to [0, 1]$, with the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1; \forall x \in X$. The values $\mu_A(x)$ and $\nu_A(x)$ represent, respectively, the membership degree and non-membership degree of the element x to the set A. For any intuitionistic fuzzy set A on the universal set X, for $x \in X$

$$\pi_A(x) = 1 - \left(\mu_A(x) + \nu_A(x)\right),$$

is called the hesitancy degree (or intuitionistic fuzzy index) of an element x in A. It is the degree of indeterminacy membership of the element x whether belonging to A or not. Obviously, $0 \leq \pi_A(x) \leq 1$ for any $x \in X$.

Particularly, $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is always valid for any fuzzy set A on the universal set X.The set of all picture fuzzy sets in X will be denoted by $PFS(X)$.

Definition 2.3 ([7,8]). A picture fuzzy set A on a universe of discourse X is of the form

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)): x \in X\},$$

where $\mu_A(x) \in [0, 1]$ is called the degree of positive membership of x in A, $\eta_A(x) \in [0, 1]$ is called the degree of neutral membership of x in A and $\nu_A(x) \in [0, 1]$ is called the degree of negative membership of x in A, and where $\mu_A(x), \eta_A(x)$ and $\nu_A(x)$ satisfy the following condition:

$$0 \leq \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1; \forall x \in X.$$

Here $1 - \left(\mu_A(x) + \eta_A(x) + \nu_A(x)\right); \forall x \in X$ is called the degree of refusal membership of x in A. The set of all picture fuzzy sets in X will be denoted by $PFS(X)$.

Definition 2.4 ([7,8]). Let $A, B \in PFS(X)$, then the subset, equality, the union, intersection and complement are defined as follows:

i. $A \subseteq B$ iff $\forall x \in X, \mu_A(x) \leq \mu_B(x), \eta_A(x) \leq \eta_B(x)$ and $\nu_A(x) \geq \nu_B(x)$;
ii. $A = B$ iff $\forall x \in X, \mu_A(x) = \mu_B(x), \eta_A(x) = \eta_B(x)$ and $\nu_A(x) = \nu_B(x)$;
iii. $A \cup B = \left\{\left(x, \max\left(\mu_A(x), \mu_B(x)\right), \min\left(\eta_A(x), \eta_B(x)\right), \min\left(\nu_A(x), \nu_B(x)\right)\right): x \in X\right\}$;
iv. $A \cap B = \left\{\left(x, \min\left(\mu_A(x), \mu_B(x)\right), \min\left(\eta_A(x), \eta_B(x)\right), \max\left(\nu_A(x), \nu_B(x)\right)\right): x \in X\right\}$;
v. $A^c = \left\{\left(x, \nu_A(x), \eta_A(x), \mu_A(x)\right): x \in X\right\}$.

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Definition 2.5. ([13]) Let $A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) : x \in X\}$ be a picture fuzzy set on X and $\alpha, \gamma, \beta \in [0,1]$, $\alpha + \gamma + \beta \leq 1$, then the upper (α, γ, β)-cut of A is given by

$$A^{(\alpha,\gamma,\beta)} = \{x \in X: \mu_A(x) \geq \alpha, \eta_A(x) \geq \gamma, \nu_A(x) \leq \beta\}$$

That is, $\alpha_{\mu_A} = \{x: \mu_A(x) \geq \alpha\}$, $\gamma_{\eta_A} = \{x: \eta_A(x) \geq \gamma\}$ and $\beta_{\nu_A} = \{x: \nu_A(x) \leq \beta\}$ are upper α, γ and β- cut of positive membership, neutral membership and negative membership of a picture fuzzy set A respectively.

Definition 2.6. ([14]) Let $\omega_1, \omega_2, \omega_3 \in [0,1]$ with $0 \leq \omega_1 + \omega_2 + \omega_3 \leq 1$. A generalized picture fuzzy number (GPFN) $\tilde{\rho}$ is a special picture fuzzy set of real numbers $\mathbb{R}$ whose membership functions $\mu_{\tilde{\rho}}(x): \mathbb{R} \longrightarrow [0, \omega_1]$, $\eta_{\tilde{\rho}}(x): \mathbb{R} \longrightarrow [0, \omega_2]$ and $\nu_{\tilde{\rho}}(x): \mathbb{R} \longrightarrow [\omega_3, 1]$ satisfy the following conditions:

i. There exist at least three real numbers x_1, x_2 and x_3 such that $\mu_{\tilde{\rho}}(x_1) = \omega_1$, $\eta_{\tilde{\rho}}(x_2) = \omega_2$ and $\nu_{\tilde{\rho}}(x_3) = \omega_3$.

ii. $\mu_{\tilde{\rho}}$ and $\eta_{\tilde{\rho}}$ are quasi concave and upper semi continuous on $\mathbb{R}$.

iii. $\nu_{\tilde{\rho}}$ is quasi convex and lower semi continuous on $\mathbb{R}$.

iv. The support of $\tilde{\rho}$ is compact.

Definition 2.7.([14]) A generalized trapezoidal picture fuzzy number (GTraPFN) $\tilde{\rho} = \langle (a, b, c, d); \omega_1, \omega_2, \omega_3 \rangle$ is a special picture fuzzy set on $\mathbb{R}$ whose positive , neutral and negative membership functions are defined as follows:

$$\mu_{\tilde{\rho}}(x) = \begin{cases} 0 & ; x < a \\ \frac{\omega_1(x-a)}{(b-a)} & ; a \leq x < b \\ \omega_1 & ; b \leq x \leq c \\ \frac{\omega_1(d-x)}{(d-c)} & ; c < x \leq d \\ 0 & ; x > d \end{cases}, \quad \eta_{\tilde{\rho}}(x) = \begin{cases} 0 & ; x < a \\ \frac{\omega_2(x-a)}{(b-a)} & ; a \leq x < b \\ \omega_2 & ; b \leq x \leq c \\ \frac{\omega_2(d-x)}{(d-c)} & ; c < x \leq d \\ 0 & ; x > d \end{cases}, \quad \nu_{\tilde{\rho}}(x) = \begin{cases} 1 & ; x < a \\ \frac{b-x+\omega_3(x-a)}{(b-a)} & ; a \leq x < b \\ \omega_3 & ; b \leq x \leq c \\ \frac{x-c+\omega_3(d-x)}{(d-c)} & ; c < x \leq d \\ 1 & ; x > d \end{cases}$$

The following **Fig. 1** is the graphical representation of the GTraPFN:

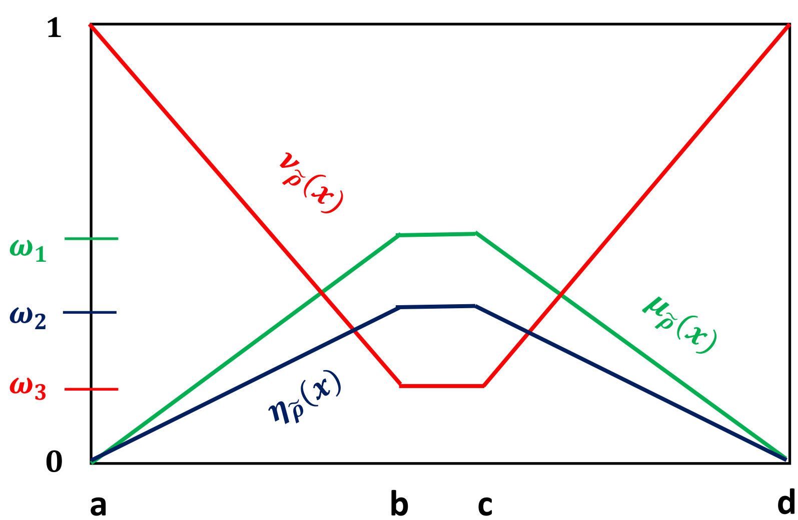

Fig. 1: Generalized TraPFN

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Definition 2.8.([15]) A generalized triangular picture fuzzy number (GTPFN) $\tilde{\rho} = \langle(a, b, c); \omega_1, \omega_2, \omega_3\rangle$ is a special picture fuzzy set on $\mathbb{R}$ whose positive, neutral and negative membership functions are defined as follows:

$$\mu_{\tilde{\rho}}(x) = \begin{cases} 0 & ; x < a \\ \frac{\omega_1(x-a)}{(b-a)} & ; a \leq x \leq b \\ \frac{\omega_1(c-x)}{(c-b)} & ; b < x \leq c \\ 0 & ; x > c \end{cases}, \eta_{\tilde{\rho}}(x) \begin{cases} 0 & ; x < a \\ \frac{\omega_2(x-a)}{(b-a)} & ; a \leq x \leq b \\ \frac{\omega_2(c-x)}{(c-b)} & ; b < x \leq c \\ 0 & ; x > c \end{cases},$$

$$\nu_{\tilde{\rho}}(x) = \begin{cases} 1 & ; x < a \\ \frac{b - x + \omega_3(x-a)}{(b-a)} & ; a \leq x \leq b \\ \frac{x - b + \omega_3(c-x)}{(c-b)} & ; b < x \leq c \\ 1 & ; x > c \end{cases}$$

The following **Fig. 2** is the graphical representation of the GTPFN:

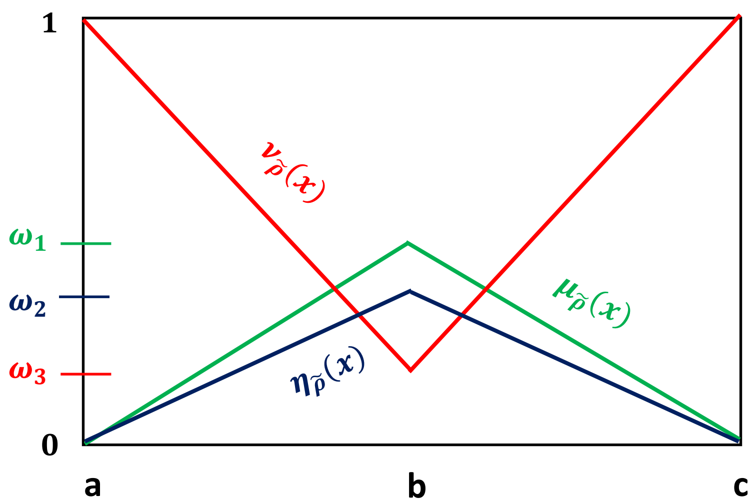

Fig. 2: Generalized TPFN

Definition 2.9. ([14]) A GTraPFN $A = \langle(a_1, a_2, a_3, a_4); \omega_1, \omega_2, \omega_3\rangle$ is said to be monotonic increasing if $a_1 \leq a_2 \leq a_3 \leq a_4$.

Definition 2.10. ([14,15]) A GTraPFN $A = \langle(a_1, a_2, a_3, a_4); \omega_1, \omega_2, \omega_3\rangle$ is called positive if $a_i > 0$, for $i = 1,2,3,4$.

Definition 2.11. ([14,15]) A GTraPFN $A = \langle(a_1, a_2, a_3, a_4); \omega_1, \omega_2, \omega_3\rangle$ is called negative if $a_i < 0$, for $i = 1,2,3,4$.

Definition 2.12. ([14,15]) A GTraPFN $A = \langle(a_1, a_2, a_3, a_4); \omega_1, \omega_2, \omega_3\rangle$ is called partial if at least one $a_i < 0$, for $i = 1,2,3,4$.

Definition 2.13. ([14]) Let $A = \langle(a, b, c, d); \omega_1, \omega_2, \omega_3\rangle$ be a GTraPFN. Then the α −cut of A is a crisp subset of X which is defined as follows:

$$A^\alpha = \{x: \mu_A(x) \geq \alpha\} = [A_1(\alpha), A_2(\alpha)] = \left[a + \frac{\alpha}{\omega_1}(b-a), d - \frac{\alpha}{\omega_1}(d-c)\right]; \alpha \in [0, \omega_1]$$

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Definition 2.14. ([14]) Let $A = \langle (a, b, c, d); \omega_1, \omega_2, \omega_3 \rangle$ be a GTraPFN. Then the γ −cut of A is a crisp subset of X which is defined as follows:

$$A^{\gamma} = \{x: \eta_A(x) \geq \gamma\} = [A_1(\gamma), A_2(\gamma)] = \left[a + \frac{\gamma}{\omega_2}(b-a), d - \frac{\gamma}{\omega_2}(d-c)\right]; \gamma \in [0, \omega_2]$$

Definition 2.15. ([14]) Let $A = \langle (a, b, c, d); \omega_1, \omega_2, \omega_3 \rangle$ be a GTraPFN. Then the β −cut of A is a crisp subset of X which is defined as follows:

$$A^{\beta} = \{x: \nu_A(x) \leq \beta\} = [A_1(\beta), A_2(\beta)] = \left[\frac{(b-\omega_3 a)-\beta(b-a)}{1-\omega_3}, \frac{\beta(d-c)-(\omega_3 d-c)}{1-\omega_3}\right]; \beta \in [\omega_3, 1]$$

Definition 2.16. ([15])A Generalized Triangular Picture Fuzzy Number $A = \langle (a_1, a_2, a_3); \omega_1, \omega_2, \omega_3 \rangle$ is called positive if $a_i > 0$, for $i = 1,2,3$.

Definition 2.17. ([15])A Generalized Triangular Picture Fuzzy Number $A = \langle (a_1, a_2, a_3); \omega_1, \omega_2, \omega_3 \rangle$ is called negative if $a_i < 0$, for $i = 1,2,3$.

Definition 2.18. ([15])A Generalized Triangular Picture Fuzzy Number $A = \langle (a_1, a_2, a_3); \omega_1, \omega_2, \omega_3 \rangle$ is called partial if at least one $a_i < 0$, for $i = 1,2,3$.

Definition 2.19. ([15])Let $A = \langle (a, b, c); \omega_1, \omega_2, \omega_3 \rangle$ be a GTPFN. Then the α −cut of A is a crisp subset of $\mathbb{R}$ which is defined as follows:

$$A^{\alpha} = \{x: \mu_A(x) \geq \alpha\} = [A_1(\alpha), A_2(\alpha)] = \left[a + \frac{\alpha(b-a)}{\omega_1}, c - \frac{\alpha(c-b)}{\omega_1}\right]; \alpha \in [0, \omega_1].$$

Definition 2.20. ([15])Let $A = \langle (a, b, c); \omega_1, \omega_2, \omega_3 \rangle$ be a GTPFN. Then the γ −cut of A is a crisp subset of $\mathbb{R}$ which is defined as follows:

$$A^{\gamma} = \{x: \eta_A(x) \geq \gamma\} = [A_1(\gamma), A_2(\gamma)] = \left[a + \frac{\gamma(b-a)}{\omega_2}, c - \frac{\gamma(c-b)}{\omega_2}\right]; \gamma \in [0, \omega_2].$$

Definition 2.21. ([15])Let $A = \langle (a, b, c); \omega_1, \omega_2, \omega_3 \rangle$ be a GTPFN. Then the β −cut of A is a crisp subset of $\mathbb{R}$ which is defined as follows:

$$A^{\beta} = \{x: \nu_A(x) \leq \beta\} = [A_1(\beta), A_2(\beta)] = \left[\frac{(b-a\omega_3)-\beta(b-a)}{1-\omega_3}, \frac{\beta(c-b)-(\omega_3 c-b)}{1-\omega_3}\right]; \beta \in [\omega_3, 1].$$

Definition 2.22. ([14]) Let $A = \langle (a, b, c, d); \omega_1, \omega_2, \omega_3 \rangle$ be a GTraPFN. Then the (α, γ, β) −cut of A is given by

$$A^{(\alpha,\gamma,\beta)} = \{[A_1(\alpha), A_2(\alpha)], [A_1(\gamma), A_2(\gamma)], [A_1(\beta), A_2(\beta)]\};$$
$$0 < \alpha + \gamma + \beta \leq 1,\ \alpha \in [0, \omega_1],\ \gamma \in [0, \omega_2], \beta \in [\omega_3, 1].$$

Definition 2.23. ([14]) Let A and B be two GTraPFNs and their corresponding (α, γ, β) −cuts are

$$A^{(\alpha,\gamma,\beta)} = \{[A_1(\alpha), A_2(\alpha)], [A_1(\gamma), A_2(\gamma)], [A_1(\beta), A_2(\beta)]\}$$
$$B^{(\alpha,\gamma,\beta)} = \{[B_1(\alpha), B_2(\alpha)], [B_1(\gamma), B_2(\gamma)], [B_1(\beta), B_2(\beta)]\}$$

for any $\alpha, \gamma, \beta \in [0,1]$ and $0 \leq \alpha + \gamma + \beta \leq 1$, then the four basic arithmetic operations such as addition, subtraction, multiplication and division are defined as follows:

Addition:

$$(A+B)^{(\alpha,\gamma,\beta)} = \{[A_1(\alpha) + B_1(\alpha), A_2(\alpha) + B_2(\alpha)], [A_1(\gamma) + B_1(\gamma), A_2(\gamma) + B_2(\gamma)], [A_1(\beta) + B_1(\beta), A_2(\beta) + B_2(\beta)]\}$$

Subtraction:

$$A^{(\alpha,\gamma,\beta)} - B^{(\alpha,\gamma,\beta)} = \{[A_1(\alpha) - B_2(\alpha), A_2(\alpha) - B_1(\alpha)], [A_1(\gamma) - B_2(\gamma), A_2(\gamma) - B_1(\gamma)], [A_1(\beta) - B_2(\beta), A_2(\beta) - B_1(\beta)]\}$$

Multiplication:

$$A^{(\alpha,\gamma,\beta)} \times B^{(\alpha,\gamma,\beta)} = \{[minM^{\alpha}, maxM^{\alpha}], [minM^{\gamma}, maxM^{\gamma}], [minM^{\beta}, maxM^{\beta}]\}$$

where $M^{\alpha} = \{A_1(\alpha).B_1(\alpha), A_1(\alpha).B_2(\alpha), A_2(\alpha).B_1(\alpha), A_2(\alpha).B_2(\alpha)\}$

$M^{\gamma} = \{A_1(\gamma).B_1(\gamma), A_1(\gamma).B_2(\gamma), A_2(\gamma).B_1(\gamma), A_2(\gamma).B_2(\gamma)\}$

$M^{\beta} = \{A_1(\beta).B_1(\beta), A_1(\beta).B_2(\beta), A_2(\beta).B_1(\beta), A_2(\beta).B_2(\beta)\}$

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Division:

$A^{(\alpha,\gamma,\beta)} \div B^{(\alpha,\gamma,\beta)} = \{[minD^{\alpha}, maxD^{\alpha}], [minD^{\gamma}, maxD^{\gamma}], [minD^{\beta}, maxD^{\beta}]\}$

where $D^{\alpha} = \left\{\frac{A_1(\alpha)}{B_1(\alpha)}, \frac{A_1(\alpha)}{B_2(\alpha)}, \frac{A_2(\alpha)}{B_1(\alpha)}, \frac{A_2(\alpha)}{B_2(\alpha)}\right\}$, $D^{\gamma} = \left\{\frac{A_1(\gamma)}{B_1(\gamma)}, \frac{A_1(\gamma)}{B_2(\gamma)}, \frac{A_2(\gamma)}{B_1(\gamma)}, \frac{A_2(\gamma)}{B_2(\gamma)}\right\}$ and

$D^{\beta} = \left\{\frac{A_1(\beta)}{B_1(\beta)}, \frac{A_1(\beta)}{B_2(\beta)}, \frac{A_2(\beta)}{B_1(\beta)}, \frac{A_2(\beta)}{B_2(\beta)}\right\}$

where $B_i(\alpha) \neq 0, B_i(\gamma) \neq 0, B_i(\beta) \neq 0\ ; i = 1,2.$

3. Arithmetic Operations of GPFNs by $(\alpha, \gamma, \beta) -$ cut Method

Proposition 3.1: Scalar multiplication of a generalized trapezoidal picture fuzzy number is also a generalized trapezoidal picture fuzzy number.

Proof: Let $A = \langle(a_1, a_2, a_3, a_4); \omega_1, \omega_2, \omega_3\rangle$ be a GTraPFN. For any scalar k , the scalar multiplication of A is given as $C^{(\alpha,\gamma,\beta)} = kA^{(\alpha,\gamma,\beta)}$, where $C^{\alpha} = [C_1(\alpha), C_2(\alpha)]$, $C^{\gamma} = [C_1(\gamma), C_2(\gamma)]$ and $C^{\beta} = [C_1(\beta), C_2(\beta)]$, where $\alpha \in [0, \omega_1]$, $\gamma \in [0, \omega_2]$ and $\beta \in [\omega_3, 1]$.

Case-1: When $k > 0$

$C^{\alpha} = [C_1(\alpha), C_2(\alpha)]$

$= k[A_1(\alpha), A_2(\alpha)]$

$= [kA_1(\alpha), kA_2(\alpha)]$

$= \left[ka_1 + k\frac{\alpha}{\omega_1}(a_2 - a_1), ka_4 - k\frac{\alpha}{\omega_1}(a_4 - a_3)\right]$

Let, $ka_1 + k\frac{\alpha}{\omega_1}(a_2 - a_1) \leq z \leq ka_4 - k\frac{\alpha}{\omega_1}(a_4 - a_3)$

Now, $ka_1 + k\frac{\alpha}{\omega_1}(a_2 - a_1) \leq z$

$\Rightarrow k\frac{\alpha}{\omega_1}(a_2 - a_1) \leq z - ka_1$

$\Rightarrow \alpha k(a_2 - a_1) \leq \omega_1(z - ka_1)$

$\Rightarrow \alpha \leq \omega_1\frac{(z - ka_1)}{k(a_2 - a_1)}$

$\Rightarrow \omega_1\frac{(z - ka_1)}{(ka_2 - ka_1)} \geq \alpha$

Let, $\mu_C^L(z) = \omega_1\frac{(z - ka_1)}{(ka_2 - ka_1)}$

Now, $\frac{d}{dz}\mu_C^L(z) = \frac{d}{dz}\omega_1\frac{(z - ka_1)}{(ka_2 - ka_1)} = \frac{\omega_1}{(ka_2 - ka_1)}\frac{d}{dz}(z - ka_1) = \frac{\omega_1}{(ka_2 - ka_1)}(1 - 0)$

$= \frac{\omega_1}{(ka_2 - ka_1)} > 0$; if $ka_2 > ka_1$

Therefore, $\mu_C^L(z)$ is an increasing function.

Also $\mu_C^L(ka_1) = \omega_1\frac{(ka_1 - ka_1)}{(ka_2 - ka_1)} = 0$, $\mu_C^L(ka_2) = \omega_1\frac{(ka_2 - ka_1)}{(ka_2 - ka_1)} = \omega_1$ and $\mu_C^L\left(\frac{ka_1 + ka_2}{2}\right) > \frac{\omega_1}{2}$.

Again, $ka_4 - k\frac{\alpha}{\omega_1}(a_4 - a_3) \geq z$

$\Rightarrow ka_4 - k\frac{\alpha}{\omega_1}(a_4 - a_3) \geq z$

$\Rightarrow -k\frac{\alpha}{\omega_1}(a_4 - a_3) \geq (z - ka_4)$

$\Rightarrow k\frac{\alpha}{\omega_1}(a_4 - a_3) \leq (ka_4 - z)$

$\Rightarrow \alpha \leq \omega_1\frac{(ka_4 - z)}{k(a_4 - a_3)}$

$\Rightarrow \omega_1\frac{(ka_4 - z)}{(ka_4 - ka_3)} \geq \alpha$

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Let, $\mu_C^R(z) = \omega_1 \frac{(ka_4 - z)}{(ka_4 - ka_3)}$

Now, $\frac{d}{dz}\mu_C^R(z) = \frac{d}{dz}\omega_1 \frac{(ka_4 - z)}{(ka_4 - ka_3)} = \frac{\omega_1}{(ka_4 - ka_3)}\frac{d}{dz}(ka_4 - z) = \frac{\omega_1}{(ka_4 - ka_3)}(0 - 1)$

$= -\frac{\omega_1}{(ka_4 - ka_3)} < 0$; if $ka_4 > ka_3$

Therefore, $\mu_C^R(z)$ is a decreasing function.

Also, $\mu_C^R(ka_4) = \omega_1 \frac{(ka_4 - ka_4)}{(ka_4 - ka_3)} = 0$, $\mu_C^R(ka_3) = \omega_1 \frac{(ka_4 - ka_3)}{(ka_4 - ka_3)} = \omega_1$ and $\mu_C^R\left(\frac{ka_3 + ka_4}{2}\right) < \frac{\omega_1}{2}$.

So the positive membership function of $C = kA$ is

$$\mu_C(z) = \begin{cases} \omega_1 \frac{(z - ka_1)}{(ka_2 - ka_1)} & ;\ ka_1 \le z \le ka_2 \\ \omega_1 & ;\ ka_2 \le z \le ka_3 \\ \omega_1 \frac{(ka_4 - z)}{(ka_4 - ka_3)} & ;\ ka_3 \le z \le ka_4 \\ 0 & ;\ otherwise \end{cases}$$

Similarly, we can find the neutral membership function.

$$\eta_C(z) = \begin{cases} \omega_2 \frac{(z - ka_1)}{(ka_2 - ka_1)} & ;\ ka_1 \le z \le ka_2 \\ \omega_2 & ;\ ka_2 \le z \le ka_3 \\ \omega_2 \frac{(ka_4 - z)}{(ka_4 - ka_3)} & ;\ ka_3 \le z \le ka_4 \\ 0 & ;\ otherwise \end{cases}$$

Now, for the negative membership function

$C^\beta = [C_1(\beta), C_2(\beta)]$

$= k[A_1(\beta), A_2(\beta)]$

$= [kA_1(\beta), kA_2(\beta)]$

$= \left[k\frac{(a_2 - \omega_3 a_1) - \beta(a_2 - a_1)}{1 - \omega_3}, k\frac{\beta(a_4 - a_3) - (\omega_3 a_4 - a_3)}{1 - \omega_3}\right]$

Let, $k\frac{(a_2 - \omega_3 a_1) - \beta(a_2 - a_1)}{1 - \omega_3} \le z \le k\frac{\beta(a_4 - a_3) - (\omega_3 a_4 - a_3)}{1 - \omega_3}$

Now, $k\frac{(a_2 - \omega_3 a_1) - \beta(a_2 - a_1)}{1 - \omega_3} \le z$

$\Rightarrow (a_2 - \omega_3 a_1) - \beta(a_2 - a_1) \le \frac{(1 - \omega_3)z}{k}$

$\Rightarrow -\beta(a_2 - a_1) \le \frac{(1 - \omega_3)z}{k} - (a_2 - \omega_3 a_1)$

$\Rightarrow -\beta \le \frac{(1 - \omega_3)z}{k(a_2 - a_1)} - \frac{(a_2 - \omega_3 a_1)}{(a_2 - a_1)}$

$\Rightarrow \beta \ge \frac{(a_2 - \omega_3 a_1)}{(a_2 - a_1)} - \frac{(1 - \omega_3)z}{k(a_2 - a_1)}$

$\Rightarrow \beta \ge \frac{k(a_2 - \omega_3 a_1) - (1 - \omega_3)z}{k(a_2 - a_1)}$

$\Rightarrow \frac{k(a_2 - \omega_3 a_1) - (1 - \omega_3)z}{(ka_2 - ka_1)} \le \beta$

Let, $\nu_C^L(z) = \frac{k(a_2 - \omega_3 a_1) - (1 - \omega_3)z}{(ka_2 - ka_1)}$

Now, $\frac{d}{dz}\nu_C^L(z) = \frac{d}{dz}\frac{k(a_2 - \omega_3 a_1) - (1 - \omega_3)z}{(ka_2 - ka_1)} = \frac{1}{(ka_2 - ka_1)}\frac{d}{dz}\{k(a_2 - \omega_3 a_1) - (1 - \omega_3)z\}$

$= \frac{1}{(ka_2 - ka_1)}\{0 - (1 - \omega_3)\} = -\frac{(1 - \omega_3)}{(ka_2 - ka_1)} < 0$; if $ka_2 > ka_1$

Therefore, $\nu_C^L(z)$ is a deceasing function.

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Also, $\nu_C^L(ka_2) = \frac{k(a_2-\omega_3 a_1)-(1-\omega_3)ka_2}{(ka_2-ka_1)} = \frac{ka_2-\omega_3 ka_1-ka_2+\omega_3 ka_2}{(ka_2-ka_1)} = \frac{-\omega_3 ka_1+\omega_3 ka_2}{(ka_2-ka_1)}$
$= \frac{\omega_3(ka_2-ka_1)}{(ka_2-ka_1)} = \omega_3$
$\nu_C^L(ka_1) = \frac{k(a_2-\omega_3 a_1)-(1-\omega_3)ka_1}{(ka_2-ka_1)} = \frac{ka_2-\omega_3 ka_1-ka_1+\omega_3 ka_1}{(ka_2-ka_1)} = \frac{(ka_2-ka_1)}{(ka_2-ka_1)} = 1$
and $\nu_C^L\left(\frac{ka_2+ka_1}{2}\right) < \frac{(1+\omega_3)}{2}$.
Again, $k\,\frac{\beta(a_4-a_3)-(\omega_3 a_4-a_3)}{1-\omega_3} \geq z$
$\Rightarrow \beta(a_4-a_3)-(\omega_3 a_4-a_3) \geq \frac{(1-\omega_3)z}{k}$
$\Rightarrow \beta(a_4-a_3) \geq \frac{(1-\omega_3)z}{k} + (\omega_3 a_4-a_3)$
$\Rightarrow \beta \geq \frac{(1-\omega_3)z}{k(a_4-a_3)} + \frac{(\omega_3 a_4-a_3)}{(a_4-a_3)}$
$\Rightarrow \beta \geq \frac{(1-\omega_3)z+k(\omega_3 a_4-a_3)}{k(a_4-a_3)}$
$\Rightarrow \frac{(1-\omega_3)z+k(\omega_3 a_4-a_3)}{(ka_4-ka_3)} \leq \beta$
Let, $\nu_C^R(z) = \frac{(1-\omega_3)z+k(\omega_3 a_4-a_3)}{(ka_4-ka_3)}$
Now, $\frac{d}{dz}\nu_C^R(z) = \frac{d}{dz}\frac{(1-\omega_3)z+k(\omega_3 a_4-a_3)}{(ka_4-ka_3)} = \frac{1}{(ka_4-ka_3)}\frac{d}{dz}\{(1-\omega_3)z+k(\omega_3 a_4-a_3)\}$
$= \frac{1}{(ka_4-ka_3)}\{(1-\omega_3)+0\} = \frac{(1-\omega_3)}{(ka_4-ka_3)} > 0$; if $ka_4 > ka_3$
Therefore, $\nu_C^R(z)$ is an increasing function.
Also, $\nu_C^R(ka_3) = \frac{(1-\omega_3)ka_3+k(\omega_3 a_4-a_3)}{(ka_4-ka_3)} = \frac{ka_3-\omega_3 ka_3+\omega_3 ka_4-ka_3}{(ka_4-ka_3)} = \frac{-\omega_3 ka_3+\omega_3 ka_4}{(ka_4-ka_3)}$
$= \frac{\omega_3(ka_4-ka_3)}{(ka_4-ka_3)} = \omega_3$
$\nu_C^R(ka_4) = \frac{(1-\omega_3)ka_4+k(\omega_3 a_4-a_3)}{(ka_4-ka_3)} = \frac{ka_4-\omega_3 ka_4+\omega_3 ka_4-ka_3}{(ka_4-ka_3)} = \frac{ka_4-ka_3}{(ka_4-ka_3)} = 1$
and $\nu_C^R\left(\frac{ka_3+ka_4}{2}\right) > \frac{(1+\omega_3)}{2}$.
So the negative membership function of $C = kA$ is

$$\nu_C(z) = \begin{cases} \frac{k(a_2-\omega_3 a_1)-(1-\omega_3)z}{(ka_2-ka_1)} & ;\ ka_1 \leq z \leq ka_2 \\ \omega_3 & ;\ ka_2 \leq z \leq ka_3 \\ \frac{(1-\omega_3)z+k(\omega_3 a_4-a_3)}{(ka_4-ka_3)} & ;\ ka_3 \leq z \leq ka_4 \\ 1 & ;\ otherwise \end{cases}$$

Case-2: When $k < 0$
$C^\alpha = [C_1(\alpha), C_2(\alpha)]$
$= k[A_1(\alpha), A_2(\alpha)]$
$= [kA_2(\alpha), kA_1(\alpha)]$
$= \left[ka_4 - k\frac{\alpha}{\omega_1}(a_4-a_3), ka_1 + k\frac{\alpha}{\omega_1}(a_2-a_1)\right]$
Let, $ka_4 - k\frac{\alpha}{\omega_1}(a_4-a_3) \leq z \leq ka_1 + k\frac{\alpha}{\omega_1}(a_2-a_1)$
Now, $ka_4 - k\frac{\alpha}{\omega_1}(a_4-a_3) \leq z$
$\Rightarrow -k\frac{\alpha}{\omega_1}(a_4-a_3) \leq z - ka_4$

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$\Rightarrow \frac{\alpha}{\omega_1}(ka_3 - ka_4) \leq z - ka_4$
$\Rightarrow \alpha \leq \omega_1 \frac{(z-ka_4)}{(ka_3-ka_4)}$
$\Rightarrow \omega_1 \frac{(z-ka_4)}{(ka_3-ka_4)} \geq \alpha$
Let, $\mu_C^L(z) = \omega_1 \frac{(z-ka_4)}{(ka_3-ka_4)}$
Now, $\frac{d}{dz}\mu_C^L(z) = \frac{d}{dz}\omega_1 \frac{(z-ka_4)}{(ka_3-ka_4)} = \frac{\omega_1}{(ka_3-ka_4)}\frac{d}{dz}(z - ka_4) = \frac{\omega_1}{(ka_3-ka_4)}(1-0)$
$= \frac{\omega_1}{(ka_3-ka_4)} > 0$; if $ka_3 > ka_4$
Therefore, $\mu_C^L(z)$ is an increasing function.
Also $\mu_C^L(ka_4) = \omega_1 \frac{(ka_4-ka_4)}{(ka_3-ka_4)} = 0$, $\mu_C^L(ka_3) = \omega_1 \frac{(ka_3-ka_4)}{(ka_3-ka_4)} = \omega_1$ and $\mu_C^L\left(\frac{ka_4+ka_3}{2}\right) > \frac{\omega_1}{2}$.
Again, $ka_1 + k\frac{\alpha}{\omega_1}(a_2 - a_1) \geq z$
$\Rightarrow k\frac{\alpha}{\omega_1}(a_2 - a_1) \geq z - ka_1$
$\Rightarrow -k\frac{\alpha}{\omega_1}(a_2 - a_1) \leq (ka_1 - z)$
$\Rightarrow \frac{\alpha}{\omega_1}(ka_1 - ka_2) \leq (ka_1 - z)$
$\Rightarrow \alpha \leq \omega_1 \frac{(ka_1-z)}{(ka_1-ka_2)}$
$\Rightarrow \omega_1 \frac{(ka_1-z)}{(ka_1-ka_2)} \geq \alpha$
Let, $\mu_C^R(z) = \omega_1 \frac{(ka_1-z)}{(ka_1-ka_2)}$
Now, $\frac{d}{dz}\mu_C^R(z) = \frac{d}{dz}\omega_1 \frac{(ka_1-z)}{(ka_1-ka_2)} = \frac{\omega_1}{(ka_1-ka_2)}\frac{d}{dz}(ka_1 - z)$
$= \frac{\omega_1}{(ka_1-ka_2)}(0-1) = -\frac{\omega_1}{(ka_1-ka_2)} < 0$; if $ka_1 > ka_2$
Therefore, $\mu_C^R(z)$ is a decreasing function.
Also, $\mu_C^R(ka_1) = \omega_1 \frac{(ka_1-ka_1)}{(ka_1-ka_2)} = 0$, $\mu_C^R(ka_2) = \omega_1 \frac{(ka_1-ka_2)}{(ka_1-ka_2)} = \omega_1$ and $\mu_C^R\left(\frac{ka_1+ka_2}{2}\right) < \frac{\omega_1}{2}$.
So the positive membership function of $C = kA$ is

$$\mu_C(z) = \begin{cases} \omega_1 \frac{(z-ka_4)}{(ka_3-ka_4)} & ;\ ka_4 \leq z \leq ka_3 \\ \omega_1 & ;\ ka_3 \leq z \leq ka_2 \\ \omega_1 \frac{(ka_1-z)}{(ka_1-ka_2)} & ;\ ka_2 \leq z \leq ka_1 \\ 0 & ;\ otherwise \end{cases}$$

Similarly, we can find the neutral membership function.

$$\eta_C(z) = \begin{cases} \omega_2 \frac{(z-ka_4)}{(ka_3-ka_4)} & ;\ ka_4 \leq z \leq ka_3 \\ \omega_2 & ;\ ka_3 \leq z \leq ka_2 \\ \omega_2 \frac{(ka_1-z)}{(ka_1-ka_2)} & ;\ ka_2 \leq z \leq ka_1 \\ 0 & ;\ otherwise \end{cases}$$

Now, for the negative membership function
$C^\beta = [C_1(\beta), C_2(\beta)]$
$= k[A_1(\beta), A_2(\beta)]$
$= [kA_2(\beta), kA_1(\beta)]$

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$= \left[k\frac{\beta(a_4-a_3)-(\omega_3a_4-a_3)}{1-\omega_3}, k\frac{(a_2-a_1\omega_3)-\beta(a_2-a_1)}{1-\omega_3}\right]$

Let, $k\frac{\beta(a_4-a_3)-(\omega_3a_4-a_3)}{1-\omega_3} \leq z \leq k\frac{(a_2-a_1\omega_3)-\beta(a_2-a_1)}{1-\omega_3}$

Now, $k\frac{\beta(a_4-a_3)-(\omega_3a_4-a_3)}{1-\omega_3} \leq z$

$\Rightarrow k\{\beta(a_4-a_3)-(\omega_3a_4-a_3)\} \leq z(1-\omega_3)$

$\Rightarrow -k\{\beta(a_4-a_3)-(\omega_3a_4-a_3)\} \geq -z(1-\omega_3)$

$\Rightarrow -k\beta(a_4-a_3)+k(\omega_3a_4-a_3) \geq -z(1-\omega_3)$

$\Rightarrow -k\beta(a_4-a_3) \geq -z(1-\omega_3)-k(\omega_3a_4-a_3)$

$\Rightarrow k\beta(a_3-a_4) \geq -z(1-\omega_3)-k(\omega_3a_4-a_3)$

$\Rightarrow \beta \geq \frac{-(1-\omega_3)z-k(\omega_3a_4-a_3)}{(ka_3-ka_4)}$

$\Rightarrow \frac{-(1-\omega_3)z-k(\omega_3a_4-a_3)}{(ka_3-ka_4)} \leq \beta$

Let, $\nu_C^L(z) = \frac{-(1-\omega_3)z-k(\omega_3a_4-a_3)}{(ka_3-ka_4)}$

Now, $\frac{d}{dz}\nu_C^L(z) = \frac{d}{dz}\frac{-(1-\omega_3)z-k(\omega_3a_4-a_3)}{(ka_3-ka_4)} = \frac{1}{(ka_3-ka_4)}\frac{d}{dz}\{-(1-\omega_3)z-k(\omega_3a_4-a_3)\}$

$= \frac{1}{(ka_3-ka_4)}(-(1-\omega_3)-0) = -\frac{(1-\omega_3)}{(ka_3-ka_4)} < 0$; if $ka_3 > ka_4$

Therefore, $\nu_C^L(z)$ is a deceasing function.

Also, $\nu_C^L(ka_3) = \frac{-(1-\omega_3)ka_3-k(\omega_3a_4-a_3)}{(ka_3-ka_4)} = \frac{-ka_3+\omega_3ka_3-k\omega_3a_4+ka_3}{(ka_3-ka_4)} = \frac{\omega_3ka_3-k\omega_3a_4}{(ka_3-ka_4)} = \omega_3$

$\nu_C^L(ka_4) = \frac{-(1-\omega_3)ka_4-k(\omega_3a_4-a_3)}{(ka_3-ka_4)} = \frac{-ka_4+\omega_3ka_4-k\omega_3a_4+ka_3}{(ka_3-ka_4)} = \frac{-ka_4+ka_3}{(ka_3-ka_4)} = \frac{(ka_3-ka_4)}{(ka_3-ka_4)} = 1$

and $\nu_C^L\left(\frac{ka_3+ka_4}{2}\right) < \frac{(1+\omega_3)}{2}$.

Again,

$k\frac{(a_2-a_1\omega_3)-\beta(a_2-a_1)}{1-\omega_3} \geq z$

$\Rightarrow k\{(a_2-a_1\omega_3)-\beta(a_2-a_1)\} \geq z(1-\omega_3)$

$\Rightarrow -k\{(a_2-a_1\omega_3)-\beta(a_2-a_1)\} \leq -z(1-\omega_3)$

$\Rightarrow -k(a_2-a_1\omega_3)+\beta(ka_2-ka_1) \leq -z(1-\omega_3)$

$\Rightarrow -\beta(ka_1-ka_2) \leq -z(1-\omega_3)+k(a_2-a_1\omega_3)$

$\Rightarrow -\beta \leq \frac{-z(1-\omega_3)+k(a_2-a_1\omega_3)}{(ka_1-ka_2)}$

$\Rightarrow \beta \geq \frac{z(1-\omega_3)-k(a_2-a_1\omega_3)}{(ka_1-ka_2)}$

$\Rightarrow \frac{z(1-\omega_3)-k(a_2-a_1\omega_3)}{(ka_1-ka_2)} \leq \beta$

Let, $\nu_C^R(z) = \frac{z(1-\omega_3)-k(a_2-a_1\omega_3)}{(ka_1-ka_2)}$

Now, $\frac{d}{dz}\nu_C^R(z) = \frac{d}{dz}\frac{z(1-\omega_3)-k(a_2-a_1\omega_3)}{(ka_1-ka_2)} = \frac{1}{(ka_1-ka_2)}\frac{d}{dz}\{z(1-\omega_3)-k(a_2-a_1\omega_3)\}$

$= \frac{1}{(ka_1-ka_2)}\{(1-\omega_3)-0\} = \frac{(1-\omega_3)}{(ka_1-ka_2)} > 0$; if $ka_1 > ka_2$

Therefore, $\nu_C^R(z)$ is an increasing function.

Also, $\nu_C^R(ka_2) = \frac{ka_2(1-\omega_3)-k(a_2-a_1\omega_3)}{(ka_1-ka_2)} = \frac{ka_2-\omega_3ka_2-ka_2+ka_1\omega_3}{(ka_1-ka_2)} = \frac{-\omega_3ka_2+ka_1\omega_3}{(ka_1-ka_2)} = \omega_3$

$\nu_C^R(ka_1) = \frac{ka_1(1-\omega_3)-k(a_2-a_1\omega_3)}{(ka_1-ka_2)} = \frac{ka_1-\omega_3ka_1-ka_2+ka_1\omega_3}{(ka_1-ka_2)} = \frac{ka_1-ka_2}{(ka_1-ka_2)} = 1$

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and $\nu_C^R\left(\frac{ka_1+ka_2}{2}\right) > \frac{(1+\omega_3)}{2}$.

The negative membership function of $C = kA$ is

$$\nu_C(z) = \begin{cases} \frac{-(1-\omega_3)z-k(\omega_3 a_4-a_3)}{(ka_3-ka_4)} & ;\ ka_4 \le z \le ka_3 \\ \omega_3 & ;\ ka_3 \le z \le ka_2 \\ \frac{z(1-\omega_3)-k(a_2-a_1\omega_3)}{(ka_1-ka_2)} & ;\ ka_2 \le z \le ka_1 \\ 1 & ;\ otherwise \end{cases}$$

Therefore, the scalar multiplication of a generalized trapezoidal picture fuzzy number is also a generalized trapezoidal picture fuzzy number.

Proposition 3.2: Division of two generalized trapezoidal picture fuzzy numbers may not be a generalized trapezoidal picture fuzzy number.

Proof: Let $A = \langle(a_1, a_2, a_3, a_4); \omega_{1a}, \omega_{2a}, \omega_{3a}\rangle$ and $B = \langle(b_1, b_2, b_3, b_4); \omega_{1b}, \omega_{2b}, \omega_{3b}\rangle$ be two positive GTraPFNs. Let $A \div B = C$, where

$C^\alpha = [C_1(\alpha), C_2(\alpha)]$, $C^\gamma = [C_1(\gamma), C_2(\gamma)]$ and $C^\beta = [C_1(\beta), C_2(\beta)]$, where $\alpha \in [0, \omega_1]$, ; $\gamma \in [0, \omega_2]$, $\beta \in [\omega_3, 1]$ and $\omega_1 = min\{\omega_{1a}, \omega_{1b}\}$, $\omega_2 = min\{\omega_{2a}, \omega_{2b}\}$ and $\omega_3 = max\{\omega_{3a}, \omega_{3b}\}$.

Now,

$C^\alpha = [C_1(\alpha), C_2(\alpha)]$

$= [A_1(\alpha), A_2(\alpha)] \div [B_1(\alpha), B_2(\alpha)]$

$= \left[\frac{A_1(\alpha)}{B_2(\alpha)}, \frac{A_2(\alpha)}{B_1(\alpha)}\right]$

$= \left[\frac{a_1+\frac{\alpha}{\omega_1}(a_2-a_1)}{b_4-\frac{\alpha}{\omega_1}(b_4-b_3)}, \frac{a_4-\frac{\alpha}{\omega_1}(a_4-a_3)}{b_1+\frac{\alpha}{\omega_1}(b_2-b_1)}\right]$

Let, $\frac{a_1+\frac{\alpha}{\omega_1}(a_2-a_1)}{b_4-\frac{\alpha}{\omega_1}(b_4-b_3)} \le z \le \frac{a_4-\frac{\alpha}{\omega_1}(a_4-a_3)}{b_1+\frac{\alpha}{\omega_1}(b_2-b_1)}$

Now, $\frac{a_1+\frac{\alpha}{\omega_1}(a_2-a_1)}{b_4-\frac{\alpha}{\omega_1}(b_4-b_3)} \le z \Rightarrow \omega_1 \frac{zb_4-a_1}{(a_2-a_1)+z(b_4-b_3)} \ge \alpha$

Let, $\mu_C^L(z) = \omega_1 \frac{zb_4-a_1}{(a_2-a_1)+z(b_4-b_3)}$

Now, $\frac{d}{dz}\mu_C^L(z) = \frac{d}{dz}\omega_1 \frac{zb_4-a_1}{(a_2-a_1)+z(b_4-b_3)} = \omega_1 \frac{a_2b_4-a_1b_3}{\{(a_2-a_1)+z(b_4-b_3)\}^2} > 0$, for $a_2b_4 > a_1b_3$ i,e $\frac{a_2}{b_3} > \frac{a_1}{b_4}$.

Therefore, $\mu_C^L(z)$ is an increasing function.

Also, $\mu_C^L\left(\frac{a_2}{b_3}\right) = \omega_1$, $\mu_C^L\left(\frac{a_1}{b_4}\right) = 0$ and $\mu_C^L\left(\frac{\frac{a_2}{b_3}+\frac{a_1}{b_4}}{2}\right) = \frac{\omega_1 b_4}{b_4+b_3} > \frac{\omega_1}{2}$ [since $b_3 < b_4$].

Again,

$\frac{a_4-\frac{\alpha}{\omega_1}(a_4-a_3)}{b_1+\frac{\alpha}{\omega_1}(b_2-b_1)} \ge z \Rightarrow \omega_1 \frac{a_4-zb_1}{(a_4-a_3)+z(b_2-b_1)} \ge \alpha$

Let $\mu_C^R(z) = \omega_1 \frac{a_4-zb_1}{(a_4-a_3)+z(b_2-b_1)}$

Now, $\frac{d}{dz}\mu_C^R(z) = \frac{d}{dz}\omega_1 \frac{a_4-zb_1}{(a_4-a_3)+z(b_2-b_1)} = \omega_1 \frac{a_3b_2-a_4b_2}{\{(a_4+b_4)-(a_3+b_3)\}^2} < 0$, for $a_3b_1 < a_4b_2$ i,e $\frac{a_3}{b_2} < \frac{a_4}{b_1}$

Therefore, $\mu_C^R(z)$ is a decreasing function.

Also, $\mu_C^R\left(\frac{a_3}{b_2}\right) = \omega_1$, $\mu_C^R\left(\frac{a_4}{b_1}\right) = 0$ and $\mu_C^R\left(\frac{\frac{a_3}{b_2}+\frac{a_4}{b_1}}{2}\right) < \frac{\omega_1}{2}$.

So the positive membership function of $C = A \div B$ is

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$$\mu_C(z)=\begin{cases}\omega_1\dfrac{zb_4-a_1}{(a_2-a_1)+z(b_4-b_3)} & ;\ \dfrac{a_1}{b_4}\le z\le\dfrac{a_2}{b_3}\\ \omega_1 & ;\ \dfrac{a_2}{b_3}\le z\le\dfrac{a_3}{b_2}\\ \omega_1\dfrac{a_4-zb_1}{(a_4-a_3)+z(b_2-b_1)} & ;\ \dfrac{a_3}{b_2}\le z\le\dfrac{a_4}{b_1}\\ 0 & ;\ otherwise\end{cases}$$

Similarly, we can find the neutral membership function.

So the neutral membership function of $C = A \div B$ is

$$\eta_C(z)=\begin{cases}\omega_2\dfrac{zb_4-a_1}{(a_2-a_1)+z(b_4-b_3)} & ;\ \dfrac{a_1}{b_4}\le z\le\dfrac{a_2}{b_3}\\ \omega_2 & ;\ \dfrac{a_2}{b_3}\le z\le\dfrac{a_3}{b_2}\\ \omega_2\dfrac{a_4-zb_1}{(a_4-a_3)+z(b_2-b_1)} & ;\ \dfrac{a_3}{b_2}\le z\le\dfrac{a_4}{b_1}\\ 0 & ;\ otherwise\end{cases}$$

Now, for the negative membership function

$C^{\beta} = [C_1(\beta), C_2(\beta)]$

$= [A_1(\beta), A_2(\beta)] \div [B_1(\beta), B_2(\beta)]$

$= \left[\dfrac{A_1(\beta)}{B_2(\beta)}, \dfrac{A_2(\beta)}{B_1(\beta)}\right]$

$= \left[\left\{\frac{(a_2-a_1\omega_3)-\beta(a_2-a_1)}{1-\omega_3}\right\} \div \left\{\frac{\beta(b_4-b_3)-(\omega_3 b_4-b_3)}{1-\omega_3}\right\}, \left\{\frac{\beta(a_4-a_3)-(\omega_3 a_4-a_3)}{1-\omega_3}\right\} \div \left\{\frac{(b_2-b_1\omega_3)-\beta(b_2-b_1)}{1-\omega_3}\right\}\right]$

$= \left[\frac{(a_2-a_1\omega_3)-\beta(a_2-a_1)}{\beta(b_4-b_3)-(\omega_3 b_4-b_3)}, \frac{\beta(a_4-a_3)-(\omega_3 a_4-a_3)}{(b_2-b_1\omega_3)-\beta(b_2-b_1)}\right]$

Let, $\frac{(a_2-a_1\omega_3)-\beta(a_2-a_1)}{\beta(b_4-b_3)-(\omega_3 b_4-b_3)} \le z < \frac{\beta(a_4-a_3)-(\omega_3 a_4-a_3)}{(b_2-b_1\omega_3)-\beta(b_2-b_1)}$

Now, $\frac{(a_2-a_1\omega_3)-\beta(a_2-a_1)}{\beta(b_4-b_3)-(\omega_3 b_4-b_3)} \le z$

$\Rightarrow (a_2-a_1\omega_3)-\beta(a_2-a_1) \le z\{\beta(b_4-b_3)-(\omega_3 b_4-b_3)\}$

$\Rightarrow (a_2-a_1\omega_3)-\beta(a_2-a_1) \le \beta z(b_4-b_3)-z(\omega_3 b_4-b_3)$

$\Rightarrow -\beta z(b_4-b_3)-\beta(a_2-a_1) \le -z(\omega_3 b_4-b_3)-(a_2-a_1\omega_3)$

$\Rightarrow -\beta\{z(b_4-b_3)+(a_2-a_1)\} \le -\{z(\omega_3 b_4-b_3)+(a_2-a_1\omega_3)\}$

$\Rightarrow \beta\{z(b_4-b_3)+(a_2-a_1)\} \ge \{z(\omega_3 b_4-b_3)+(a_2-a_1\omega_3)\}$

$\Rightarrow \beta \ge \frac{\{z(\omega_3 b_4-b_3)+(a_2-a_1\omega_3)\}}{\{z(b_4-b_3)+(a_2-a_1)\}}$

$\Rightarrow \frac{\{z(\omega_3 b_4-b_3)+(a_2-a_1\omega_3)\}}{\{z(b_4-b_3)+(a_2-a_1)\}} \le \beta$

Let, $\nu_C^L(z) = \frac{\{z(\omega_3 b_4-b_3)+(a_2-a_1\omega_3)\}}{\{z(b_4-b_3)+(a_2-a_1)\}}$

Now, $\frac{d}{dz}\nu_C^L(z) = \frac{d}{dz}\frac{\{z(\omega_3 b_4-b_3)+(a_2-a_1\omega_3)\}}{\{z(b_4-b_3)+(a_2-a_1)\}}$

$$= \frac{\{z(b_4-b_3)+(a_2-a_1)\}(\omega_3 b_4-b_3)-\{z(\omega_3 b_4-b_3)+(a_2-a_1\omega_3)\}(b_4-b_3)}{\{z(b_4-b_3)+(a_2-a_1)\}^2} > 0$$

Therefore, $\nu_C^L(z)$ is an increasing function.

Also, $\nu_C^L\left(\frac{a_2}{b_3}\right) = \omega_3$, $\mu_C^L\left(\frac{a_1}{b_4}\right) = 0$ and $\nu_C^L\left(\frac{\frac{a_2}{b_3}+\frac{a_1}{b_4}}{2}\right) = \frac{\omega_1 b_4}{b_4+b_3} > \frac{1+\omega_3}{2}$ [since $b_3 < b_4$].

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Again,

$\frac{\beta(a_4-a_3)-(\omega_3 a_4-a_3)}{(b_2-b_1\omega_3)-\beta(b_2-b_1)} \geq z$

$\Rightarrow \beta(a_4-a_3)-(\omega_3 a_4-a_3) \geq z\{(b_2-b_1\omega_3)-\beta(b_2-b_1)\}$

$\Rightarrow \beta z(b_2-b_1)+\beta(a_4-a_3) \geq z(b_2-b_1\omega_3)+(\omega_3 a_4-a_3)$

$\Rightarrow \beta \geq \frac{z(b_2-b_1\omega_3)+(\omega_3 a_4-a_3)}{z(b_2-b_1)+(a_4-a_3)}$

$\Rightarrow \frac{z(b_2-b_1\omega_3)+(\omega_3 a_4-a_3)}{z(b_2-b_1)+(a_4-a_3)} \leq \beta$

Let, $\nu_C^R(z) = \frac{z(b_2-b_1\omega_3)+(\omega_3 a_4-a_3)}{z(b_2-b_1)+(a_4-a_3)}$

Now, $\frac{d}{dz}\nu_C^R(z) = \frac{d}{dz}\frac{z(b_2-b_1\omega_3)+(\omega_3 a_4-a_3)}{z(b_2-b_1)+(a_4-a_3)}$

$= \frac{\{z(b_2-b_1)+(a_4-a_3)\}(b_2-b_1\omega_3)-\{z(b_2-b_1\omega_3)+(\omega_3 a_4-a_3)\}(b_2-b_1)}{\{z(b_2-b_1)+(a_4-a_3)\}^2} < 0$, for $a_2 b_1 < a_3 b_2$ i,e $\frac{a_3}{b_2} < \frac{a_4}{b_1}$.

Therefore, $\nu_C^R(z)$ is a decreasing function.

Also, $\nu_C^R\left(\frac{a_3}{b_2}\right) = \omega_3$, $\nu_C^R\left(\frac{a_4}{b_1}\right) = 1$ and $\nu_C^R\left(\frac{\frac{a_3}{b_2}+\frac{a_4}{b_1}}{2}\right) < \frac{1+\omega_3}{2}$.

So the negative membership function of $C = A \div B$ is

$$\nu_C(z) = \begin{cases} \frac{\{z(\omega_3 b_3-b_2)+(a_2-a_1\omega_3)\}}{\{z(b_3-b_2)+(a_2-a_1)\}} & ; \frac{a_1}{b_4} \leq z \leq \frac{a_2}{b_3} \\ \omega_3 & ; \frac{a_2}{b_3} \leq z \leq \frac{a_3}{b_2} \\ \frac{z(b_2-b_1\omega_3)+(\omega_3 a_3-a_2)}{z(b_2-b_1)+(a_3-a_2)} & ; \frac{a_3}{b_2} \leq z \leq \frac{a_4}{b_1} \\ 1 & ; otherwise \end{cases}$$

Proposition 3.3: Multiplication of two generalized trapezoidal picture fuzzy numbers may not be a generalized trapezoidal picture fuzzy number.

Proof: Let $A = \langle(a_1,a_2,a_3,a_4);\omega_{1a},\omega_{2a},\omega_{3a}\rangle$ and $B = \langle(b_1,b_2,b_3,b_4);\omega_{1b},\omega_{2b},\omega_{3b}\rangle$ be two positive GTraPFNs. Let $A \times B = C$, where $C^\alpha = [C_1(\alpha), C_2(\alpha)]$, $C^\gamma = [C_1(\gamma), C_2(\gamma)]$ and $C^\beta = [C_1(\beta), C_2(\beta)]$, where $\alpha \in [0,\omega_1]$, ; $\gamma \in [0,\omega_2]$, $\beta \in [\omega_3, 1]$ and $\omega_1 = min\{\omega_{1a},\omega_{1b}\}$, $\omega_2 = min\{\omega_{2a},\omega_{2b}\}$ and $\omega_3 = max\{\omega_{3a},\omega_{3b}\}$.

Now,

$C^\alpha = [C_1(\alpha), C_2(\alpha)]$

$= [A_1(\alpha), A_2(\alpha)].[B_1(\alpha), B_2(\alpha)]$

$= [A_1(\alpha)B_1(\alpha), A_2(\alpha)B_2(\alpha)]$

$= \left[\left\{a_1+\frac{\alpha}{\omega_1}(a_2-a_1)\right\}\left\{b_1+\frac{\alpha}{\omega_1}(b_2-b_1)\right\}, \left\{a_4-\frac{\alpha}{\omega_1}(a_4-a_3)\right\}\left\{b_4-\frac{\alpha}{\omega_1}(b_4-b_3)\right\}\right]$

$= \left[\frac{\alpha^2}{\omega_1^2}(a_2-a_1)(b_2-b_1)+\frac{\alpha}{\omega_1}\{a_1(b_2-b_1)+b_1(a_2-a_1)\}+a_1b_1, \frac{\alpha^2}{\omega_1^2}(a_4-a_3)(b_4-b_3) - \frac{\alpha}{\omega_1}\{a_4(b_4-b_3)+b_4(a_4-a_3)\}+a_4b_4\right]$

Let, $\frac{\alpha^2}{\omega_1^2}(a_2-a_1)(b_2-b_1)+\frac{\alpha}{\omega_1}\{a_1(b_2-b_1)+b_1(a_2-a_1)\}+a_1b_1 \leq z \leq \frac{\alpha^2}{\omega_1^2}(a_4-a_3)(b_4-b_3)-\frac{\alpha}{\omega_1}\{a_4(b_4-b_3)+b_4(a_4-a_3)\}+a_4b_4$

Let, $P_1 = (a_2-a_1)(b_2-b_1)$ and $Q_1 = a_1(b_2-b_1)+b_1(a_2-a_1)$

Now $\frac{\alpha^2}{\omega_1^2}P_1+\frac{\alpha}{\omega_1}Q_1+a_1b_1 \leq z$

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$$\Rightarrow \frac{\alpha^2}{\omega_1^2}P_1 + \frac{\alpha}{\omega_1}Q_1 + a_1b_1 - z \le 0$$
$$\Rightarrow \frac{-Q_1 - \sqrt{Q_1^2 - 4P_1(a_1b_1 - z)}}{2P_1} \le \frac{\alpha}{\omega_1} \le \frac{-Q_1 + \sqrt{Q_1^2 - 4P_1(a_1b_1 - z)}}{2P_1}$$
$$\Rightarrow \omega_1\frac{-Q_1 - \sqrt{Q_1^2 - 4P_1(a_1b_1 - z)}}{2P_1} \le \alpha \le \omega_1\frac{-Q_1 + \sqrt{Q_1^2 - 4P_1(a_1b_1 - z)}}{2P_1}$$

Let, $\mu_C^L(z) = \omega_1\frac{-Q_1 + \sqrt{Q_1^2 - 4P_1(a_1b_1 - z)}}{2P_1}$

Now, $\frac{d}{dz}\mu_C^L(z) = \frac{d}{dz}\omega_1\frac{-Q_1 + \sqrt{Q_1^2 - 4P_1(a_1b_1 - z)}}{2P_1}$

$$= \frac{\omega_1}{2P_1}\frac{d}{dz}\left\{-Q_1 + \sqrt{Q_1^2 - 4P_1(a_1b_1 - z)}\right\}$$
$$= \frac{\omega_1}{2P_1}\left\{-0 + \frac{1}{2}\left(Q_1^2 - 4P_1(a_1b_1 - z)\right)^{-\frac{1}{2}}\frac{d}{dz}\left(Q_1^2 - 4P_1(a_1b_1 - z)\right)\right\}$$
$$= \frac{\omega_1}{2P_1}\left\{\frac{1}{2}\left(Q_1^2 - 4P_1(a_1b_1 - z)\right)^{-\frac{1}{2}}\left(0 - 4P_1(0 - 1)\right)\right\}$$
$$= \frac{\omega_1}{4P_1}\left\{\frac{4P_1}{\sqrt{\left(Q_1^2 - 4P_1(a_1b_1 - z)\right)}}\right\} = \frac{\omega_1}{\sqrt{\left(Q_1^2 - 4P_1(a_1b_1 - z)\right)}}$$
$$= \frac{\omega_1}{\sqrt{\left(\{a_1(b_2 - b_1) + b_1(a_2 - a_1)\}^2 - 4(a_2 - a_1)(b_2 - b_1)(a_1b_1 - z)\right)}} > 0$$

Therefore, $\mu_C^L(z)$ is an increasing function.

Also, $\mu_C^L(a_1b_1) = \omega_1\frac{-Q_1 + \sqrt{Q_1^2 - 4P_1(a_1b_1 - a_1b_1)}}{2P_1} = \omega_1\frac{-Q_1 + \sqrt{Q_1^2 - 0}}{2P_1} = \omega_1\frac{-Q_1 + Q_1}{2P_1} = 0$

$\mu_C^L(a_2b_2) = \omega_1$ and $\mu_C^L\left(\frac{a_1b_1 + a_2b_2}{2}\right) > \frac{\omega_1}{2}$

Again,

Let, $P_2 = (a_4 - a_3)(b_4 - b_3)$ and $Q_2 = a_4(b_4 - b_3) + b_4(a_4 - a_3)$, then

$$\frac{\alpha^2}{\omega_1^2}P_2 + \frac{\alpha}{\omega_1}Q_2 + a_4b_4 \ge z$$
$$\Rightarrow \frac{\alpha^2}{\omega_1^2}P_2 + \frac{\alpha}{\omega_1}Q_2 + a_4b_4 - z \ge 0$$
$$\Rightarrow \frac{-Q_2 - \sqrt{Q_2^2 - 4P_2(a_4b_4 - z)}}{2P_2} \ge \frac{\alpha}{\omega_1} \ge \frac{-Q_2 + \sqrt{Q_2^2 - 4P_2(a_4b_4 - z)}}{2P_2}$$
$$\Rightarrow \omega_1\frac{-Q_2 - \sqrt{Q_2^2 - 4P_2(a_4b_4 - z)}}{2P_2} \ge \alpha \ge \omega_1\frac{-Q_2 + \sqrt{Q_2^2 - 4P_2(a_4b_4 - z)}}{2P_2}$$
$$\Rightarrow \omega_1\frac{Q_2 + \sqrt{Q_2^2 - 4P_2(a_4b_4 - z)}}{2P_2} \le \alpha \le \omega_1\frac{Q_2 - \sqrt{Q_2^2 - 4P_2(a_4b_4 - z)}}{2P_2}$$

Let $\mu_C^R(z) = \omega_1\frac{Q_2 - \sqrt{Q_2^2 - 4P_2(a_4b_4 - z)}}{2P_2}$

Now, $\frac{d}{dz}\mu_C^R(z) = \frac{d}{dz}\omega_1\frac{Q_2 - \sqrt{Q_2^2 - 4P_2(a_4b_4 - z)}}{2P_2}$

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$$= \frac{\omega_1}{2P_2}\frac{d}{dz}\left\{Q_2 - \sqrt{Q_2^2 - 4P_2(a_4b_4 - z)}\right\}$$
$$= \frac{\omega_1}{2P_2}\left\{-0 - \frac{1}{2}\left(Q_2^2 - 4P_2(a_4b_4 - z)\right)^{-\frac{1}{2}}\frac{d}{dz}\left(Q_2^2 - 4P_2(a_4b_4 - z)\right)\right\}$$
$$= \frac{\omega_1}{2P_1}\left\{-\frac{1}{2}\left(Q_2^2 - 4P_2(a_4b_4 - z)\right)^{-\frac{1}{2}}\left(0 - 4P_2(0 - 1)\right)\right\}$$
$$= \frac{\omega_1}{4P_2}\left\{\frac{-4P_2}{\sqrt{\left(Q_2^2 - 4P_2(a_4b_4 - z)\right)}}\right\} = -\frac{\omega_1}{\sqrt{\left(Q_2^2 - 4P_2(a_4b_4 - z)\right)}}$$
$$= -\frac{\omega_1}{\sqrt{\left(\{a_4(b_4 - b_3) + b_4(a_4 - a_3)\}^2 - 4(a_4 - a_3)(b_4 - b_3)(a_4b_4 - z)\right)}} < 0$$

Therefore, $\mu_C^R(z)$ is a decreasing function.

Also $\mu_C^R(a_4b_4) = \omega_1 \frac{Q_2 - \sqrt{Q_2^2 - 4P_2(a_4b_4 - a_4b_4)}}{2P_2} = \omega_1 \frac{Q_2 - \sqrt{Q_2^2 - 0}}{2P_2} = \omega_1 \frac{Q_2 - Q_2}{2P_2} = 0$ and

$\mu_C^R(a_3b_3) = \omega_1$ and $\mu_C^R\left(\frac{a_3b_3 + a_4b_4}{2}\right) < \frac{\omega_1}{2}$

So the positive membership function of $C = A \times B$ is

$$\mu_C(z) = \begin{cases} \omega_1 \frac{-Q_1 + \sqrt{Q_1^2 - 4P_1(a_1b_1 - z)}}{2P_1} & ;\ a_1b_1 \le z \le a_2b_2 \\ \omega_1 & ;\ a_2b_2 \le z \le a_3b_3 \\ \omega_1 \frac{Q_2 - \sqrt{Q_2^2 - 4P_2(a_4b_4 - z)}}{2P_2} & ;\ a_3b_3 \le z \le a_4b_4 \\ 0 & ;\ otherwise \end{cases}$$

Similarly, we can find the neutral membership function.

So the neutral membership function of $C = A \times B$ is

$$\eta_C(z) = \begin{cases} \omega_2 \frac{-Q_1 + \sqrt{Q_1^2 - 4P_1(a_1b_1 - z)}}{2P_1} & ;\ a_1b_1 \le z \le a_2b_2 \\ \omega_2 & ;\ a_2b_2 \le z \le a_3b_3 \\ \omega_2 \frac{Q_2 - \sqrt{Q_2^2 - 4P_2(a_4b_4 - z)}}{2P_2} & ;\ a_3b_3 \le z \le a_4b_4 \\ 0 & ;\ otherwise \end{cases}$$

Now, for the negative membership function

$C^\beta = [C_1(\beta), C_2(\beta)]$

$= [A_1(\beta), A_2(\beta)].[B_1(\beta), B_2(\beta)]$

$= [A_1(\beta)B_1(\beta), A_2(\beta)B_2(\beta)]$

$$= \left[\frac{(a_2 - \omega_3a_1) - \beta(a_2 - a_1)}{1 - \omega_3} \times \frac{(b_2 - \omega_3b_1) - \beta(b_2 - b_1)}{1 - \omega_3}, \frac{\beta(a_4 - a_3) - (\omega_3a_4 - a_3)}{1 - \omega_3} \times \frac{\beta(b_4 - b_3) - (\omega_3b_4 - b_3)}{1 - \omega_3}\right]$$

$$= \left[\frac{\beta^2}{(1 - \omega_3)^2}(a_2 - a_1)(b_2 - b_1) - \frac{\beta}{(1 - \omega_3)}\{a_2(b_2 - b_1) + b_2(a_2 - a_1)\} + a_2b_2, \frac{\beta^2}{(1 - \omega_3)^2}(a_4 - a_3)(b_4 - b_3) + \frac{\beta}{(1 - \omega_3)}\{a_3(b_4 - b_3) + b_3(a_4 - a_3)\} + a_3b_3\right]$$

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Let, $\frac{\beta^2}{(1-\omega_3)^2}\ (a_2-a_1)(b_2-b_1)-\frac{\beta}{(1-\omega_3)}\{a_2(b_2-b_1)+b_2(a_2-a_1)\}+a_2b_2\le z\le$ $\frac{\beta^2}{(1-\omega_3)^2}\ (a_4-a_3)(b_4-b_3)+\frac{\beta}{(1-\omega_3)}\{a_3(b_4-b_3)+b_3(a_4-a_3)\}+a_3b_3$

Let $P_1'=(a_2-a_1)(b_2-b_1)$ and $Q_1'=a_2(b_2-b_1)+b_2(a_2-a_1)$

Now,

$$\frac{\beta^2}{(1-\omega_3)^2}\ P_1'-\frac{\beta}{(1-\omega_3)}Q_1'+a_2b_2\le z$$

$$\Rightarrow \frac{\beta^2}{(1-\omega_3)^2}\ P_1'-\frac{\beta}{(1-\omega_3)}Q_1'+a_2b_2-z\le 0$$

$$\Rightarrow \frac{Q_1'-\sqrt{{Q_1'}^2-4P_1'(a_2b_2-z)}}{2P_1'}\le\frac{\beta}{(1-\omega_3)}\le\frac{Q_1'+\sqrt{{Q_1'}^2-4P_1'(a_2b_2-z)}}{2P_1'}$$

$$\Rightarrow (1-\omega_3)\frac{Q_1'-\sqrt{{Q_1'}^2-4P_1'(a_2b_2-z)}}{2P_1'}\le\beta\le(1-\omega_3)\frac{Q_1'+\sqrt{{Q_1'}^2-4P_1'(a_2b_2-z)}}{2P_1'}$$

Let, $\nu_C^L(z)=(1-\omega_3)\frac{Q_1'-\sqrt{{Q_1'}^2-4P_1'(a_2b_2-z)}}{2P_1'}$

Now, $\frac{d}{dz}\nu_C^L(z)=\frac{d}{dz}(1-\omega_3)\frac{Q_1'-\sqrt{{Q_1'}^2-4P_1'(a_2b_2-z)}}{2P_1'}$

$$=-\frac{(1-\omega_3)}{\sqrt{(\{a_2(b_2-b_1)+b_2(a_2-a_1)\}^2-4(a_2-a_1)(b_2-b_1)(a_2b_2-z))}}<0$$

Therefore, $\nu_C^L(z)$ is a decreasing function.

Also, $\nu_C^L(a_1b_1)=\omega_3$, $\nu_C^L(a_2b_2)=0$ and $\nu_C^L\left(\frac{a_2b_2+a_1b_1}{2}\right)<\frac{(1+\omega_3)}{2}$

Again,

$$\frac{\beta^2}{(1-\omega_3)^2}\ (a_4-a_3)(b_4-b_3)+\frac{\beta}{(1-\omega_3)}\{a_3(b_4-b_3)+b_3(a_4-a_3)\}+a_3b_3\ge z$$

Let, $P_2'=(a_4-a_3)(b_4-b_3)$ and $Q_2'=a_3(b_4-b_3)+b_3(a_4-a_3)$, then

Now,

$$\frac{\beta^2}{(1-\omega_3)^2}\ P_2'+\frac{\beta}{(1-\omega_3)}Q_2'+a_3b_3\ge z$$

$$\Rightarrow \frac{\beta^2}{(1-\omega_3)^2}\ P_2'+\frac{\beta}{(1-\omega_3)}Q_2'+a_3b_3-z\ge 0$$

$$\Rightarrow \frac{-Q_2'-\sqrt{{Q_2'}^2-4P_2'(a_3b_3-z)}}{2P_2'}\ge\frac{\beta}{(1-\omega_3)}\ge\frac{-Q_2'+\sqrt{{Q_2'}^2-4P_2'(a_3b_3-z)}}{2P_2'}$$

$$\Rightarrow (1-\omega_3)\frac{-Q_2'-\sqrt{{Q_2'}^2-4P_2'(a_3b_3-z)}}{2P_2'}\ge\beta\ge(1-\omega_3)\frac{-Q_2'+\sqrt{{Q_2'}^2-4P_2'(a_3b_3-z)}}{2P_2'}$$

Let, $\nu_C^R(z)=(1-\omega_3)\frac{-Q_2'+\sqrt{{Q_2'}^2-4P_2'(a_3b_3-z)}}{2P_2'}$

Now, $\frac{d}{dz}\nu_C^R(z)=\frac{d}{dz}(1-\omega_3)\frac{-Q_2'+\sqrt{{Q_2'}^2-4P_2'(a_3b_3-z)}}{2P_2'}$

$$=\frac{(1-\omega_3)}{\sqrt{(\{a_3(b_4-b_3)+b_3(a_4-a_3)\}^2-4(a_4-a_3)(b_4-b_3)(a_3b_3-z))}}>0$$

Therefore, $\nu_C^R(z)$ is an increasing function.

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Also, $\nu_C^R(a_3b_3)=0$, $\nu_C^R(a_4b_4)=\omega_3$ and $\nu_C^R\left(\frac{a_3b_3+a_4b_4}{2}\right)>\frac{(1+\omega_3)}{2}$

So the negative membership function of $C=A\times B$ is

$$\nu_C(z)=\begin{cases}(1-\omega_3)\frac{Q_1'-\sqrt{Q_1'^2-4P_1'(a_2b_2-z)}}{2P_1'} & ;\ a_1b_1\le z\le a_2b_2\\ \omega_3 & ;\ a_2b_2\le z\le a_3b_3\\ (1-\omega_3)\frac{-Q_2'+\sqrt{Q_2'^2-4P_2'(a_3b_3-z)}}{2P_2'} & ;\ a_3b_3\le z\le a_4b_4\\ 1 & ;\ otherwise\end{cases}$$

Proposition 3.4: Scalar multiplication of a generalized triangular picture fuzzy number is also a generalized triangular picture fuzzy number.

Proof: Let $A=\langle(a_1,a_2,a_3);\omega_1,\omega_2,\omega_3\rangle$ be a GTPFN . For any scalar k, the scalar multiplication of A is

$C^{(\alpha,\gamma,\beta)}=kA^{(\alpha,\gamma,\beta)}$, where $C^\alpha=[C_1(\alpha),C_2(\alpha)]$, $C^\gamma=[C_1(\gamma),C_2(\gamma)]$ and $C^\beta=[C_1(\beta),C_2(\beta)]$ and $\alpha\in[0,\omega_1]$, $\gamma\in[0,\omega_2]$, $\beta\in[\omega_3,1]$.

Case-1: When $k>0$

$C^\alpha=[C_1(\alpha),C_2(\alpha)]$
$=k[A_1(\alpha),A_2(\alpha)]$
$=[kA_1(\alpha),kA_2(\alpha)]$
$=\left[ka_1+k\frac{\alpha}{\omega_1}(a_2-a_1),ka_3-k\frac{\alpha}{\omega_1}(a_3-a_2)\right]$

Let, $ka_1+k\frac{\alpha}{\omega_1}(a_2-a_1)\le z\le ka_3-k\frac{\alpha}{\omega_1}(a_3-a_2)$

Now, $ka_1+k\frac{\alpha}{\omega_1}(a_2-a_1)\le z$

$\Rightarrow k\frac{\alpha}{\omega_1}(a_2-a_1)\le z-ka_1$

$\Rightarrow\le\omega_1(z-ka_1)$

$\Rightarrow \alpha\le\omega_1\frac{(z-ka_1)}{k(a_2-a_1)}$

$\Rightarrow \omega_1\frac{(z-ka_1)}{(ka_2-ka_1)}\ge\alpha$

Let, $\mu_C^L(z)=\omega_1\frac{(z-ka_1)}{(ka_2-ka_1)}$

Now, $\frac{d}{dz}\mu_C^L(z)=\frac{d}{dz}\omega_1\frac{(z-ka_1)}{(ka_2-ka_1)}$

$=\frac{\omega_1}{(ka_2-ka_1)}\frac{d}{dz}(z-ka_1)=\frac{\omega_1}{(ka_2-ka_1)}(1-0)=\frac{\omega_1}{(ka_2-ka_1)}>0$; if $ka_2>ka_1$

Therefore, $\mu_C^L(z)$ is an increasing function.

Also, $\mu_C^L(ka_1)=\omega_1\frac{(ka_1-ka_1)}{(ka_2-ka_1)}=0$ and $\mu_C^L(ka_2)=\omega_1\frac{(ka_2-ka_1)}{(ka_2-ka_1)}=\omega_1$

Again, $ka_3-k\frac{\alpha}{\omega_1}(a_3-a_2)z$

$\Rightarrow ka_3-k\frac{\alpha}{\omega_1}(a_3-a_2)\ge z$

$\Rightarrow -k\frac{\alpha}{\omega_1}(a_3-a_2)\ge(z-ka_3)$

$\Rightarrow k\frac{\alpha}{\omega_1}(a_3-a_2)\le(ka_3-z)$

$\Rightarrow \alpha\le\omega_1\frac{(ka_3-z)}{k(a_3-a_2)}$

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$\Rightarrow \omega_1 \frac{(ka_3-z)}{k(a_3-a_2)} \geq \alpha$

Let, $\mu_C^R(z) = \omega_1 \frac{(ka_3-z)}{(ka_3-ka_2)}$

Now, $\frac{d}{dz}\mu_C^R(z) = \frac{d}{dz}\omega_1 \frac{(ka_3-z)}{(ka_3-ka_2)}$

$= \frac{\omega_1}{(ka_3-ka_2)}\frac{d}{dz}(ka_3 - z) = \frac{\omega_1}{(ka_3-ka_2)}(0-1) = -\frac{\omega_1}{(ka_3-ka_2)} < 0$; if $ka_3 > ka_2$

Therefore, $\mu_C^L(z)$ is a decreasing function.

Also, $\mu_C^R(ka_3) = \omega_1 \frac{(ka_3-ka_3)}{(ka_3-ka_2)} = 0$ and $\mu_C^R(ka_2) = \omega_1 \frac{(ka_3-ka_2)}{(ka_3-ka_2)} = \omega_1$

So the positive membership function of $C = kA$ is

$$\mu_C(z) = \begin{cases} \omega_1 \frac{(z-ka_1)}{(ka_2-ka_1)} & ; \ ka_1 \leq z \leq ka_2 \\ \omega_1 \frac{(ka_3-z)}{(ka_3-ka_2)} & ; \ ka_2 \leq z \leq ka_3 \\ 0 & ; \ otherwise. \end{cases}$$

In similar manner, we can find the neutral membership of $C = kA$ is

$$\eta_C(z) = \begin{cases} \omega_2 \frac{(z-ka_1)}{(ka_2-ka_1)} & ; \ ka_1 \leq z \leq ka_2 \\ \omega_2 \frac{(ka_3-z)}{(ka_3-ka_2)} & ; \ ka_2 \leq z \leq ka_3 \\ 0 & ; \ otherwise. \end{cases}$$

Now, for the negative membership function

$C^\beta = [C_1(\beta), C_2(\beta)]$

$= k[A_1(\beta), A_2(\beta)]$

$= [kA_1(\beta), kA_2(\beta)]$

$= \left[k\frac{(a_2-\omega_3 a_1)-\beta(a_2-a_1)}{1-\omega_3}, k\frac{\beta(a_3-a_2)-(\omega_3 a_3-a_2)}{1-\omega_3}\right]$

Let, $k\frac{(a_2-\omega_3 a_1)-\beta(a_2-a_1)}{1-\omega_3} \leq z \leq k\frac{\beta(a_3-a_2)-(\omega_3 a_3-a_2)}{1-\omega_3}$

Now, $k\frac{(a_2-\omega_3 a_1)-\beta(a_2-a_1)}{1-\omega_3} \leq z$

$\Rightarrow (a_2 - \omega_3 a_1) - \beta(a_2 - a_1) \leq \frac{(1-\omega_3)z}{k}$

$\Rightarrow -\beta(a_2 - a_1) \leq \frac{(1-\omega_3)z}{k} - (a_2 - \omega_3 a_1)$

$\Rightarrow -\beta \leq \frac{(1-\omega_3)z}{k(a_2-a_1)} - \frac{(a_2-\omega_3 a_1)}{(a_2-a_1)}$

$\Rightarrow \beta \geq \frac{(a_2-\omega_3 a_1)}{(a_2-a_1)} - \frac{(1-\omega_3)z}{k(a_2-a_1)}$

$\Rightarrow \beta \geq \frac{k(a_2-\omega_3 a_1)-(1-\omega_3)z}{k(a_2-a_1)}$

$\Rightarrow \frac{k(a_2-\omega_3 a_1)-(1-\omega_3)z}{(ka_2-ka_1)} \leq \beta$

Let, $\nu_C^L(z) = \frac{k(a_2-\omega_3 a_1)-(1-\omega_3)z}{(ka_2-ka_1)}$

Now, $\frac{d}{dz}\nu_C^L(z) = \frac{d}{dz}\frac{k(a_2-\omega_3 a_1)-(1-\omega_3)z}{(ka_2-ka_1)} = \frac{1}{(ka_2-ka_1)}\frac{d}{dz}\{k(a_2 - \omega_3 a_1) - (1 - \omega_3)z\}$

$= \frac{1}{(ka_2-ka_1)}\{0 - (1 - \omega_3)\} = -\frac{(1-\omega_3)}{(ka_2-ka_1)} < 0$; if $ka_2 > ka_1$

Therefore, $\nu_C^L(z)$ is a deceasing function.

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Also, $\nu_C^L(ka_2) = \frac{k(a_2-\omega_3 a_1)-(1-\omega_3)ka_2}{(ka_2-ka_1)} = \frac{ka_2-\omega_3 ka_1-ka_2+\omega_3 ka_2}{(ka_2-ka_1)}$

$= \frac{-\omega_3 ka_1+\omega_3 ka_2}{(ka_2-ka_1)} = \frac{\omega_3(ka_2-ka_1)}{(ka_2-ka_1)} = \omega_3$

$\nu_C^L(ka_1) = \frac{k(a_2-\omega_3 a_1)-(1-\omega_3)ka_1}{(ka_2-ka_1)} = \frac{ka_2-\omega_3 ka_1-ka_1+\omega_3 ka_1}{(ka_2-ka_1)} = \frac{(ka_2-ka_1)}{(ka_2-ka_1)} = 1$

Again, $k\frac{\beta(a_3-a_2)-(\omega_3 a_3-a_2)}{1-\omega_3} \geq z$

$\Rightarrow \beta(a_3-a_2)-(\omega_3 a_3-a_2) \geq \frac{(1-\omega_3)z}{k}$

$\Rightarrow \beta(a_3-a_2) \geq \frac{(1-\omega_3)z}{k} + (\omega_3 a_3-a_2)$

$\Rightarrow \beta \geq \frac{(1-\omega_3)z}{k(a_3-a_2)} + \frac{(\omega_3 a_3-a_2)}{(a_3-a_2)}$

$\Rightarrow \beta \geq \frac{(1-\omega_3)z+k(\omega_3 a_3-a_2)}{k(a_3-a_2)}$

$\Rightarrow \frac{(1-\omega_3)z+k(\omega_3 a_3-a_2)}{k(a_3-a_2)} \leq \beta$

Let, $\nu_C^R(z) = \frac{(1-\omega_3)z+k(\omega_3 a_3-a_2)}{(ka_3-ka_2)}$

Now, $\frac{d}{dz}\nu_C^R(z) = \frac{d}{dz}\frac{(1-\omega_3)z+k(\omega_3 a_3-a_2)}{(ka_3-ka_2)} = \frac{1}{(ka_3-ka_2)}\frac{d}{dz}\{(1-\omega_3)z+k(\omega_3 a_3-a_2)\}$

$= \frac{1}{(ka_3-ka_2)}\{(1-\omega_3)+0\} = \frac{(1-\omega_3)}{(ka_3-ka_2)} > 0$; if $ka_3 > ka_2$

Therefore, $\nu_C^R(z)$ is an increasing function.

Also, $\nu_C^R(ka_3) = \frac{(1-\omega_3)ka_3+k(\omega_3 a_3-a_2)}{(ka_3-ka_2)} = \frac{ka_3-\omega_3 ka_3+\omega_3 ka_3-ka_2}{(ka_3-ka_2)} = \frac{ka_3-ka_2}{(ka_3-ka_2)} = 1$

$\nu_C^R(ka_2) = \frac{(1-\omega_3)ka_2+k(\omega_3 a_3-a_2)}{(ka_3-ka_2)} = \frac{ka_2-\omega_3 ka_2+\omega_3 ka_3-ka_2}{(ka_3-ka_2)} = \frac{-\omega_3 ka_2+\omega_3 ka_3}{(ka_3-ka_2)} = \frac{\omega_3(ka_3-ka_2)}{(ka_3-ka_2)} = \omega_3$

So the negative membership function of $C = kA$ is

$$\nu_C(z) = \begin{cases} \frac{k(a_2-\omega_3 a_1)-(1-\omega_3)z}{(ka_2-ka_1)} & ;\ ka_1 \leq z \leq ka_2 \\ \frac{(1-\omega_3)z+k(\omega_3 a_3-a_2)}{(ka_3-ka_2)} & ;\ ka_2 \leq z \leq ka_3 \\ 1 & ;\ otherwise. \end{cases}$$

Therefore, the scalar multiplication of two GTPFNs is also a GTPFN.

Case-2: When $k < 0$

$C^\alpha = [C_1(\alpha), C_2(\alpha)]$

$= k[A_1(\alpha), A_2(\alpha)]$

$= [kA_2(\alpha), kA_1(\alpha)]$

$= \left[ka_3 - k\frac{\alpha}{\omega_1}(a_3-a_2), ka_1 + k\frac{\alpha}{\omega_1}(a_2-a_1)\right]$

Let, $ka_3 - k\frac{\alpha}{\omega_1}(a_3-a_2) \leq z \leq ka_1 + k\frac{\alpha}{\omega_1}(a_2-a_1)$

Now, $ka_3 - k\frac{\alpha}{\omega_1}(a_3-a_2) \leq z$

$\Rightarrow -k\frac{\alpha}{\omega_1}(a_3-a_2) \leq z - ka_3$

$\Rightarrow \frac{\alpha}{\omega_1}(ka_2-ka_3) \leq z - ka_3$

$\Rightarrow \alpha \leq \omega_1\frac{(z-ka_3)}{(ka_2-ka_3)}$

$\Rightarrow \omega_1\frac{(z-ka_3)}{(ka_2-ka_3)} \geq \alpha$

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Let, $\mu_C^L(z) = \omega_1 \frac{(z-ka_3)}{(ka_2-ka_3)}$
Now, $\frac{d}{dz}\mu_C^L(z) = \frac{d}{dz}\omega_1 \frac{(z-ka_3)}{(ka_2-ka_3)} = \frac{\omega_1}{(ka_2-ka_3)}\frac{d}{dz}(z-ka_3)$
$= \frac{\omega_1}{(ka_2-ka_3)}(1-0) = \frac{\omega_1}{(ka_2-ka_3)} > 0$; if $ka_2 > ka_3$
Therefore, $\mu_C^L(z)$ is an increasing function.
Also $\mu_C^L(ka_3) = \omega_1 \frac{(ka_3-ka_3)}{(ka_2-ka_3)} = 0$ and $\mu_C^L(ka_2) = \omega_1 \frac{(ka_2-ka_3)}{(ka_2-ka_3)} = \omega_1$
Again, $ka_1 + k\frac{\alpha}{\omega_1}(a_2-a_1) \geq z$
$\Rightarrow k\frac{\alpha}{\omega_1}(a_2-a_1) \geq z - ka_1$
$\Rightarrow -k\frac{\alpha}{\omega_1}(a_2-a_1) \leq (ka_1 - z)$
$\Rightarrow \frac{\alpha}{\omega_1}(ka_1-ka_2) \leq (ka_1 - z)$
$\Rightarrow \alpha \leq \omega_1 \frac{(ka_1-z)}{(ka_1-ka_2)}$
$\Rightarrow \omega_1 \frac{(ka_1-z)}{(ka_1-ka_2)} \geq \alpha$
Let, $\mu_C^R(z) = \omega_1 \frac{(ka_1-z)}{(ka_1-ka_2)}$
Now, $\frac{d}{dz}\mu_C^R(z) = \frac{d}{dz}\omega_1 \frac{(ka_1-z)}{(ka_1-ka_2)} = \frac{\omega_1}{(ka_1-ka_2)}\frac{d}{dz}(ka_1-z)$
$= \frac{\omega_1}{(ka_1-ka_2)}(0-1) = -\frac{\omega_1}{(ka_1-ka_2)} < 0$; if $ka_1 > ka_2$
Therefore, $\mu_C^R(z)$ is a decreasing function.
Also, $\mu_C^R(ka_1) = \omega_1 \frac{(ka_1-ka_1)}{(ka_1-ka_2)} = 0$ and $\mu_C^R(ka_2) = \omega_1 \frac{(ka_1-ka_2)}{(ka_1-ka_2)} = \omega_1$
So the positive membership function of $C = kA$ is

$$\mu_C(z) = \begin{cases} \omega_1 \frac{(z-ka_3)}{(ka_2-ka_3)} & ;\ ka_3 \leq z \leq ka_2 \\ \omega_1 \frac{(ka_1-z)}{(ka_1-ka_2)} & ;\ ka_2 \leq z \leq ka_1 \\ 0 & ;\ otherwise. \end{cases}$$

In similar manner, we can find the neutral membership of $C = kA$ is

$$\eta_C(z) = \begin{cases} \omega_2 \frac{(z-ka_3)}{(ka_2-ka_3)} & ;\ ka_3 \leq z \leq ka_2 \\ \omega_2 \frac{(ka_1-z)}{(ka_1-ka_2)} & ;\ ka_2 \leq z \leq ka_1 \\ 0 & ;\ otherwise. \end{cases}$$

Now, for the negative membership function
$C^\beta = [C_1(\beta), C_2(\beta)]$
$= k[A_1(\beta), A_2(\beta)]$
$= [kA_2(\beta), kA_1(\beta)]$
$= \left[k\frac{\beta(a_3-a_2)-(\omega_3 a_3-a_2)}{1-\omega_3}, k\frac{(a_2-a_1\omega_3)-\beta(a_2-a_1)}{1-\omega_3}\right]$
Let, $k\frac{\beta(a_3-a_2)-(\omega_3 a_3-a_2)}{1-\omega_3} \leq z \leq k\frac{(a_2-a_1\omega_3)-\beta(a_2-a_1)}{1-\omega_3}$
Now, $k\frac{\beta(a_3-a_2)-(\omega_3 a_3-a_2)}{1-\omega_3} \leq z$
$\Rightarrow k\{\beta(a_3-a_2)-(\omega_3 a_3-a_2)\} \leq z(1-\omega_3)$

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$\Rightarrow -k\{\beta(a_3 - a_2) - (\omega_3 a_3 - a_2)\} \geq -z(1-\omega_3)$
$\Rightarrow -k\beta(a_3 - a_2) + k(\omega_3 a_3 - a_2) \geq -z(1-\omega_3)$
$\Rightarrow -k\beta(a_3 - a_2) \geq -z(1-\omega_3) - k(\omega_3 a_3 - a_2)$
$\Rightarrow k\beta(a_2 - a_3) \geq -z(1-\omega_3) - k(\omega_3 a_3 - a_2)$
$\Rightarrow \beta \geq \frac{-(1-\omega_3)z - k(\omega_3 a_3 - a_2)}{(ka_2 - ka_3)}$
$\Rightarrow \frac{-(1-\omega_3)z - k(\omega_3 a_3 - a_2)}{(ka_2 - ka_3)} \leq \beta$
Let, $\nu_C^L(z) = \frac{-(1-\omega_3)z - k(\omega_3 a_3 - a_2)}{(ka_2 - ka_3)}$
Now, $\frac{d}{dz}\nu_C^L(z) = \frac{d}{dz}\frac{-(1-\omega_3)z - k(\omega_3 a_3 - a_2)}{(ka_2 - ka_3)} = \frac{1}{(ka_2 - ka_3)}\frac{d}{dz}\{-(1-\omega_3)z - k(\omega_3 a_3 - a_2)\}$
$= \frac{1}{(ka_2 - ka_3)}(-(1-\omega_3) - 0) = -\frac{(1-\omega_3)}{(ka_2 - ka_3)} < 0$; if $ka_2 > ka_3$
Therefore, $\nu_C^L(z)$ is a deceasing function.
Also, $\nu_C^L(ka_2) = \frac{-(1-\omega_3)ka_2 - k(\omega_3 a_3 - a_2)}{(ka_2 - ka_3)} = \frac{-ka_2 + \omega_3 ka_2 - k\omega_3 a_3 + ka_2}{(ka_2 - ka_3)}$
$= \frac{\omega_3 ka_2 - k\omega_3 a_3}{(ka_2 - ka_3)} = \frac{\omega_3(ka_2 - ka_3)}{(ka_2 - ka_3)} = \omega_3$ and
$\nu_C^L(ka_3) = \frac{-(1-\omega_3)ka_3 - k(\omega_3 a_3 - a_2)}{(ka_2 - ka_3)} = \frac{-ka_3 + \omega_3 ka_3 - k\omega_3 a_3 + ka_2}{(ka_2 - ka_3)} = \frac{-ka_3 + ka_2}{(ka_2 - ka_3)} = \frac{(ka_2 - ka_3)}{(ka_2 - ka_3)} = 1$
Again, $k\frac{(a_2 - a_1\omega_3) - \beta(a_2 - a_1)}{1-\omega_3} \geq z$
$\Rightarrow k\{(a_2 - a_1\omega_3) - \beta(a_2 - a_1)\} \geq z(1-\omega_3)$
$\Rightarrow -k\{(a_2 - a_1\omega_3) - \beta(a_2 - a_1)\} \leq -z(1-\omega_3)$
$\Rightarrow -k(a_2 - a_1\omega_3) + \beta(ka_2 - ka_1) \leq -z(1-\omega_3)$
$\Rightarrow -\beta(ka_1 - ka_2) \leq -z(1-\omega_3) + k(a_2 - a_1\omega_3)$
$\Rightarrow -\beta \leq \frac{-z(1-\omega_3) + k(a_2 - a_1\omega_3)}{(ka_1 - ka_2)}$
$\Rightarrow \beta \geq \frac{z(1-\omega_3) - k(a_2 - a_1\omega_3)}{(ka_1 - ka_2)}$
$\Rightarrow \frac{z(1-\omega_3) - k(a_2 - a_1\omega_3)}{(ka_1 - ka_2)} \leq \beta$
Let, $\nu_C^R(z) = \frac{z(1-\omega_3) - k(a_2 - a_1\omega_3)}{(ka_1 - ka_2)}$
Now, $\frac{d}{dz}\nu_C^R(z) = \frac{d}{dz}\frac{z(1-\omega_3) - k(a_2 - a_1\omega_3)}{(ka_1 - ka_2)} = \frac{1}{(ka_1 - ka_2)}\frac{d}{dz}\{z(1-\omega_3) - k(a_2 - a_1\omega_3)\}$
$= \frac{1}{(ka_1 - ka_2)}\{(1-\omega_3) - 0\} = \frac{(1-\omega_3)}{(ka_1 - ka_2)} > 0$; if $ka_1 > ka_2$
Therefore, $\nu_C^R(z)$ is an increasing function.
Also, $\nu_C^R(ka_2) = \frac{ka_2(1-\omega_3) - k(a_2 - a_1\omega_3)}{(ka_1 - ka_2)} = \frac{ka_2 - \omega_3 ka_2 - ka_2 + ka_1\omega_3}{(ka_1 - ka_2)}$
$= \frac{-\omega_3 ka_2 + ka_1\omega_3}{(ka_1 - ka_2)} = \frac{\omega_3(ka_1 - ka_2)}{(ka_1 - ka_2)} = \omega_3$ and
$\nu_C^R(ka_1) = \frac{ka_1(1-\omega_3) - k(a_2 - a_1\omega_3)}{(ka_1 - ka_2)} = \frac{ka_1 - \omega_3 ka_1 - ka_2 + ka_1\omega_3}{(ka_1 - ka_2)} = \frac{ka_1 - ka_2}{(ka_1 - ka_2)} = 1$
So the negative membership function of $C = kA$ is

$$\nu_C(z) = \begin{cases} \frac{-(1-\omega_3)z - k(\omega_3 a_3 - a_2)}{(ka_2 - ka_3)} & ;\ ka_3 \leq z \leq ka_2 \\ \frac{z(1-\omega_3) - k(a_2 - a_1\omega_3)}{(ka_1 - ka_2)} & ;\ ka_2 \leq z \leq ka_1 \\ 1 & ;\ otherwise. \end{cases}$$

Therefore, the scalar multiplication of two GTPFNs is also a GTPFN.

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Proposition 3.5: Division of two generalized triangular picture fuzzy numbers may not be a generalized triangular picture fuzzy number.

Proof: Let $A = \langle (a_1, a_2, a_3); \omega_{1a}, \omega_{2a}, \omega_{3a} \rangle$ and $B = \langle (b_1, b_2, b_3); \omega_{1b}, \omega_{2b}, \omega_{3b} \rangle$ be two positive GTPFNs. Let, $C = A \div B$, where $C^{\alpha} = [C_1(\alpha), C_2(\alpha)]$, $C^{\gamma} = [C_1(\gamma), C_2(\gamma)]$ and $C^{\beta} = [C_1(\beta), C_2(\beta)]$, where $\alpha \in [0, \omega_1]$, ; $\gamma \in [0, \omega_2]$, $\beta \in [\omega_3, 1]$ and $\omega_1 = min\{\omega_{1a}, \omega_{1b}\}$, $\omega_2 = min\{\omega_{2a}, \omega_{2b}\}$ and $\omega_3 = max\{\omega_{3a}, \omega_{3b}\}$.

Now,

$C^{\alpha} = [C_1(\alpha), C_2(\alpha)]$

$= [A_1(\alpha), A_2(\alpha)] \div [B_1(\alpha), B_2(\alpha)]$

$= \left[\frac{A_1(\alpha)}{B_2(\alpha)}, \frac{A_2(\alpha)}{B_1(\alpha)}\right] = \left[\frac{a_1+\frac{\alpha}{\omega_1}(a_2-a_1)}{b_3-\frac{\alpha}{\omega_1}(b_3-b_2)}, \frac{a_3-\frac{\alpha}{\omega_1}(a_3-a_2)}{b_1+\frac{\alpha}{\omega_1}(b_2-b_1)}\right]$

Let, $\frac{a_1+\frac{\alpha}{\omega_1}(a_2-a_1)}{b_3-\frac{\alpha}{\omega_1}(b_3-b_2)} \leq z \leq \frac{a_3-\frac{\alpha}{\omega_1}(a_3-a_2)}{b_1+\frac{\alpha}{\omega_1}(b_2-b_1)}$

Now, $\frac{a_1+\frac{\alpha}{\omega_1}(a_2-a_1)}{b_3-\frac{\alpha}{\omega_1}(b_3-b_2)} \leq z \Rightarrow \omega_1 \frac{zb_3-a_1}{(a_2-a_1)+z(b_3-b_2)} \geq \alpha$

Let. $\mu_C^L(z) = \omega_1 \frac{zb_3-a_1}{(a_2-a_1)+z(b_3-b_2)}$

Now, $\frac{d}{dz}\mu_C^L(z) = \frac{d}{dz}\omega_1 \frac{zb_3-a_1}{(a_2-a_1)+z(b_3-b_2)} = \omega_1 \frac{a_2b_3-a_1b_2}{\{(a_2-a_1)+z(b_3-b_2)\}^2} > 0$, for $a_2b_3 > a_1b_2$ i,e $\frac{a_2}{b_2} > \frac{a_1}{b_3}$.

Therefore. $\mu_C^L(z)$ is an increasing function.

Also, $\mu_C^L\left(\frac{a_2}{b_2}\right) = \omega_1$, $\mu_C^L\left(\frac{a_1}{b_3}\right) = 0$ and $\mu_C^L\left(\frac{\frac{a_2}{b_2}+\frac{a_1}{b_3}}{2}\right) = \frac{\omega_1 b_3}{b_3+b_2} > \frac{\omega_1}{2}$ [since $b_2 < b_3$].

Again, $\frac{a_3-\frac{\alpha}{\omega_1}(a_3-a_2)}{b_1+\frac{\alpha}{\omega_1}(b_2-b_1)} \geq z \Rightarrow \omega_1 \frac{a_3-zb_1}{(a_3-a_2)+z(b_2-b_1)} \geq \alpha$

Let, $\mu_C^R(z) = \omega_1 \frac{a_3-zb_1}{(a_3-a_2)+z(b_2-b_1)}$

Now, $\frac{d}{dz}\mu_C^R(z) = \frac{d}{dz}\omega_1 \frac{a_3-zb_1}{(a_3-a_2)+z(b_2-b_1)} = \omega_1 \frac{a_2b_2-a_3b_2}{\{(a_3-a_2)+z(b_2-b_1)\}^2} < 0$, for $a_2b_1 < a_3b_2$ i,e $\frac{a_2}{b_2} < \frac{a_3}{b_1}$.

Therefore, $\mu_C^R(z)$ is a decreasing function.

Also, $\mu_C^R\left(\frac{a_2}{b_2}\right) = \omega_1$, $\mu_C^R\left(\frac{a_3}{b_1}\right) = 0$ and $\mu_C^R\left(\frac{\frac{a_2}{b_2}+\frac{a_3}{b_1}}{2}\right) < \frac{\omega_1}{2}$.

So the positive membership function of $C = A \div B$ is

$$\mu_C(z) = \begin{cases} \omega_1 \frac{zb_3-a_1}{(a_2-a_1)+z(b_3-b_2)} & ; \frac{a_1}{b_3} \leq z \leq \frac{a_2}{b_2} \\ \omega_1 & ; z = \frac{a_2}{b_2} \\ \omega_1 \frac{a_3-zb_1}{(a_3-a_2)+z(b_2-b_1)} & ; \frac{a_2}{b_2} \leq z \leq \frac{a_3}{b_1} \\ 0 & ; otherwise \end{cases}$$

Similarly, we can find the neutral membership function.

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$$\eta_C(z) = \begin{cases} \omega_2 \frac{zb_3 - a_1}{(a_2-a_1)+z(b_3-b_2)} & ; \frac{a_1}{b_3} \le z \le \frac{a_2}{b_2} \\ \omega_2 & ; z = \frac{a_2}{b_2} \\ \omega_2 \frac{a_3 - zb_1}{(a_3-a_2)+z(b_2-b_1)} & ; \frac{a_2}{b_2} \le z \le \frac{a_3}{b_1} \\ 0 & ; otherwise \end{cases}$$

Now, for the negative membership function

$C^\beta = [C_1(\beta), C_2(\beta)]$

$= [A_1(\beta), A_2(\beta)] \div [B_1(\beta), B_2(\beta)]$

$= \left[\frac{A_1(\beta)}{B_2(\beta)}, \frac{A_2(\beta)}{B_1(\beta)}\right]$

$= \left[\left\{\frac{(a_2-a_1\omega_3)-\beta(a_2-a_1)}{1-\omega_3}\right\} \div \left\{\frac{\beta(b_3-b_2)-(\omega_3 b_3-b_2)}{1-\omega_3}\right\}, \left\{\frac{\beta(a_3-a_2)-(\omega_3 a_3-a_2)}{1-\omega_3}\right\} \div \left\{\frac{(b_2-b_1\omega_3)-\beta(b_2-b_1)}{1-\omega_3}\right\}\right]$

$= \left[\frac{(a_2-a_1\omega_3)-\beta(a_2-a_1)}{\beta(b_3-b_2)-(\omega_3 b_3-b_2)}, \frac{\beta(a_3-a_2)-(\omega_3 a_3-a_2)}{(b_2-b_1\omega_3)-\beta(b_2-b_1)}\right]$

Let $\frac{(a_2-a_1\omega_3)-\beta(a_2-a_1)}{\beta(b_3-b_2)-(\omega_3 b_3-b_2)} \le z < \frac{\beta(a_3-a_2)-(\omega_3 a_3-a_2)}{(b_2-b_1\omega_3)-\beta(b_2-b_1)}$

Now, $\frac{(a_2-a_1\omega_3)-\beta(a_2-a_1)}{\beta(b_3-b_2)-(\omega_3 b_3-b_2)} \le z$

$\Rightarrow (a_2 - a_1\omega_3) - \beta(a_2 - a_1) \le z\{\beta(b_3 - b_2) - (\omega_3 b_3 - b_2)\}$

$\Rightarrow (a_2 - a_1\omega_3) - \beta(a_2 - a_1) \le \beta z(b_3 - b_2) - z(\omega_3 b_3 - b_2)$

$\Rightarrow -\beta z(b_3 - b_2) - \beta(a_2 - a_1) \le -z(\omega_3 b_3 - b_2) - (a_2 - a_1\omega_3)$

$\Rightarrow -\beta\{z(b_3 - b_2) + (a_2 - a_1)\} \le -\{z(\omega_3 b_3 - b_2) + (a_2 - a_1\omega_3)\}$

$\Rightarrow \beta\{z(b_3 - b_2) + (a_2 - a_1)\} \ge \{z(\omega_3 b_3 - b_2) + (a_2 - a_1\omega_3)\}$

$\Rightarrow \beta \ge \frac{\{z(\omega_3 b_3-b_2)+(a_2-a_1\omega_3)\}}{\{z(b_3-b_2)+(a_2-a_1)\}}$

$\Rightarrow \frac{\{z(\omega_3 b_3-b_2)+(a_2-a_1\omega_3)\}}{\{z(b_3-b_2)+(a_2-a_1)\}} \le \beta$

Let $\nu_C^L(z) = \frac{\{z(\omega_3 b_3-b_2)+(a_2-a_1\omega_3)\}}{\{z(b_3-b_2)+(a_2-a_1)\}}$

Now, $\frac{d}{dz}\nu_C^L(z) = \frac{d}{dz}\frac{\{z(\omega_3 b_3-b_2)+(a_2-a_1\omega_3)\}}{\{z(b_3-b_2)+(a_2-a_1)\}}$

$= \frac{\{z(b_3-b_2)+(a_2-a_1)\}(\omega_3 b_3-b_2)-\{z(\omega_3 b_3-b_2)+(a_2-a_1\omega_3)\}(b_3-b_2)}{\{z(b_3-b_2)+(a_2-a_1)\}^2} > 0$

Therefore $\nu_C^L(z)$ is an increasing function.

Also, $\nu_C^L\left(\frac{a_2}{b_2}\right) = \omega_3$, $\mu_C^L\left(\frac{a_1}{b_3}\right) = 0$ and $\nu_C^L\left(\frac{\frac{a_2}{b_2}+\frac{a_1}{b_3}}{2}\right) = \frac{\omega_1 b_3}{b_3+b_2} > \frac{1+\omega_3}{2}$ [since $b_2 < b_3$].

Again,

$\frac{\beta(a_3-a_2)-(\omega_3 a_3-a_2)}{(b_2-b_1\omega_3)-\beta(b_2-b_1)} \ge z$

$\Rightarrow \beta(a_3 - a_2) - (\omega_3 a_3 - a_2) \ge z\{(b_2 - b_1\omega_3) - \beta(b_2 - b_1)\}$

$\Rightarrow \beta z(b_2 - b_1) + \beta(a_3 - a_2) \ge z(b_2 - b_1\omega_3) + (\omega_3 a_3 - a_2)$

$\Rightarrow \beta \ge \frac{z(b_2-b_1\omega_3)+(\omega_3 a_3-a_2)}{z(b_2-b_1)+(a_3-a_2)}$

$\Rightarrow \frac{z(b_2-b_1\omega_3)+(\omega_3 a_3-a_2)}{z(b_2-b_1)+(a_3-a_2)} \le \beta$

Let, $\nu_C^R(z) = \frac{z(b_2-b_1\omega_3)+(\omega_3 a_3-a_2)}{z(b_2-b_1)+(a_3-a_2)}$

Now, $\frac{d}{dz}\nu_C^R(z) = \frac{d}{dz}\frac{z(b_2-b_1\omega_3)+(\omega_3 a_3-a_2)}{z(b_2-b_1)+(a_3-a_2)}$

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$= \frac{\{z(b_2-b_1)+(a_3-a_2)\}(b_2-b_1\omega_3)-\{z(b_2-b_1\omega_3)+(\omega_3 a_3-a_2)\}(b_2-b_1)}{\{z(b_2-b_1)+(a_3-a_2)\}^2} < 0$, for $a_2 b_1 < a_3 b_2$ i,e $\frac{a_2}{b_2} < \frac{a_3}{b_1}$.
Therefore $\nu_C^R(z)$ is a decreasing function.
Also, $\nu_C^R\left(\frac{a_2}{b_2}\right) = \omega_3$, $\nu_C^R\left(\frac{a_3}{b_1}\right) = 1$ and $\nu_C^R\left(\frac{\frac{a_2}{b_2}+\frac{a_3}{b_1}}{2}\right) < \frac{1+\omega_1}{2}$.
So the positive membership function of $C = A \div B$ is

$$\nu_C(z) = \begin{cases} \frac{\{z(\omega_3 b_3-b_2)+(a_2-a_1\omega_3)\}}{\{z(b_3-b_2)+(a_2-a_1)\}} & ; \frac{a_1}{b_3} \le z \le \frac{a_2}{b_2} \\ \omega_3 & ; z = \frac{a_2}{b_2} \\ \frac{z(b_2-b_1\omega_3)+(\omega_3 a_3-a_2)}{z(b_2-b_1)+(a_3-a_2)} & ; \frac{a_2}{b_2} \le z \le \frac{a_3}{b_1} \\ 1 & ; otherwise \end{cases}$$

Proposition 3.6: Multiplication of two generalized triangular picture fuzzy numbers may not be a generalized triangular picture fuzzy number.
Proof: Let $A = \langle(a_1, a_2, a_3); \omega_{1a}, \omega_{2a}, \omega_{3a}\rangle$ and $B = \langle(b_1, b_2, b_3); \omega_{1b}, \omega_{2b}, \omega_{3b}\rangle$ be two pisitive GTPFNs. Let $A \times B = C$, where $C^\alpha = [C_1(\alpha), C_2(\alpha)], C^\gamma = [C_1(\gamma), C_2(\gamma)]$ and $C^\beta = [C_1(\beta), C_2(\beta)]$, where $\alpha \in [0, \omega_1]$, ; $\gamma \in [0, \omega_2]$, $\beta \in [\omega_3, 1]$ and $\omega_1 = min\{\omega_{1a}, \omega_{1b}\}$, $\omega_2 = min\{\omega_{2a}, \omega_{2b}\}$ and $\omega_3 = max\{\omega_{3a}, \omega_{3b}\}$.
Now,

$$C^\alpha = [C_1(\alpha), C_2(\alpha)]$$
$$= [A_1(\alpha), A_2(\alpha)].[B_1(\alpha), B_2(\alpha)]$$
$$= [A_1(\alpha)B_1(\alpha), A_2(\alpha)B_2(\alpha)]$$
$$= \left[\left\{a_1 + \frac{\alpha}{\omega_1}(a_2-a_1)\right\}\left\{b_1 + \frac{\alpha}{\omega_1}(b_2-b_1)\right\}, \left\{a_3 - \frac{\alpha}{\omega_1}(a_3-a_2)\right\}\left\{b_3 - \frac{\alpha}{\omega_1}(b_3-b_2)\right\}\right]$$
$$= \left[\frac{\alpha^2}{\omega_1^2}(a_2-a_1)(b_2-b_1) + \frac{\alpha}{\omega_1}\{a_1(b_2-b_1)+b_1(a_2-a_1)\} + a_1 b_1, \frac{\alpha^2}{\omega_1^2}(a_3-a_2)(b_3-b_2) - \frac{\alpha}{\omega_1}\{a_3(b_3-b_2)+b_3(a_3-a_2)\} + a_3 b_3\right]$$

Let, $\frac{\alpha^2}{\omega_1^2}(a_2-a_1)(b_2-b_1) + \frac{\alpha}{\omega_1}\{a_1(b_2-b_1)+b_1(a_2-a_1)\} + a_1 b_1 \le z \le \frac{\alpha^2}{\omega_1^2}(a_3-a_2)(b_3-b_2) - \frac{\alpha}{\omega_1}\{a_3(b_3-b_2)+b_3(a_3-a_2)\} + a_3 b_3$

$$\frac{\alpha^2}{\omega_1^2}(a_2-a_1)(b_2-b_1) + \frac{\alpha}{\omega_1}\{a_1(b_2-b_1)+b_1(a_2-a_1)\} + a_1 b_1 \le z$$

Let, $P_1 = (a_2-a_1)(b_2-b_1)$ and $Q_1 = a_1(b_2-b_1)+b_1(a_2-a_1)$
Now, $\frac{\alpha^2}{\omega_1^2}P_1 + \frac{\alpha}{\omega_1}Q_1 + a_1 b_1 \le z$

$$\Rightarrow \frac{\alpha^2}{\omega_1^2}P_1 + \frac{\alpha}{\omega_1}Q_1 + a_1 b_1 - z \le 0$$
$$\Rightarrow \frac{-Q_1-\sqrt{Q_1^2-4P_1(a_1b_1-z)}}{2P_1} \le \frac{\alpha}{\omega_1} \le \frac{-Q_1+\sqrt{Q_1^2-4P_1(a_1b_1-z)}}{2P_1}$$
$$\Rightarrow \omega_1\frac{-Q_1-\sqrt{Q_1^2-4P_1(a_1b_1-z)}}{2P_1} \le \alpha \le \omega_1\frac{-Q_1+\sqrt{Q_1^2-4P_1(a_1b_1-z)}}{2P_1}$$

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Let, $\mu_C^L(z) = \omega_1 \frac{-Q_1+\sqrt{Q_1^2-4P_1(a_1b_1-z)}}{2P_1}$

Now, $\frac{d}{dz}\mu_C^L(z) = \frac{d}{dz}\omega_1 \frac{-Q_1+\sqrt{Q_1^2-4P_1(a_1b_1-z)}}{2P_1} = \frac{\omega_1}{2P_1}\frac{d}{dz}\left\{-Q_1 + \sqrt{Q_1^2 - 4P_1(a_1b_1 - z)}\right\}$

$= \frac{\omega_1}{2P_1}\left\{-0 + \frac{1}{2}\left(Q_1^2 - 4P_1(a_1b_1 - z)\right)^{-\frac{1}{2}}\frac{d}{dz}\left(Q_1^2 - 4P_1(a_1b_1 - z)\right)\right\}$

$= \frac{\omega_1}{2P_1}\left\{\frac{1}{2}\left(Q_1^2 - 4P_1(a_1b_1 - z)\right)^{-\frac{1}{2}}\left(0 - 4P_1(0 - 1)\right)\right\}$

$= \frac{\omega_1}{4P_1}\left\{\frac{4P_1}{\sqrt{\left(Q_1^2-4P_1(a_1b_1-z)\right)}}\right\} = \frac{\omega_1}{\sqrt{\left(Q_1^2-4P_1(a_1b_1-z)\right)}}$

$= \frac{\omega_1}{\sqrt{\left(\{a_1(b_2-b_1)+b_1(a_2-a_1)\}^2-4(a_2-a_1)(b_2-b_1)(a_1b_1-z)\right)}} > 0$

Therefore, $\mu_C^L(z)$ is an increasing function.

Also, $\mu_C^L(a_1b_1) = \omega_1 \frac{-Q_1+\sqrt{Q_1^2-4P_1(a_1b_1-a_1b_1)}}{2P_1} = \omega_1 \frac{-Q_1+\sqrt{Q_1^2-0}}{2P_1} = \omega_1 \frac{-Q_1+Q_1}{2P_1} = 0$, $\mu_C^L(a_2b_2) = \omega_1$ and $\mu_C^L\left(\frac{a_1b_1+a_2b_2}{2}\right) > \frac{\omega_1}{2}$

Again, $\frac{\alpha^2}{\omega_1^2}(a_3 - a_2)(b_3 - b_2) - \frac{\alpha}{\omega_1}\{a_3(b_3 - b_2) + b_3(a_3 - a_2)\} + a_3b_3 \geq z$

Let, $P_2 = (a_3 - a_2)(b_3 - b_2)$ and $Q_2 = a_3(b_3 - b_2) + b_3(a_3 - a_2)$, then

$\frac{\alpha^2}{\omega_1^2}P_2 + \frac{\alpha}{\omega_1}Q_2 + a_3b_3 \geq z$

$\Rightarrow \frac{\alpha^2}{\omega_1^2}P_2 + \frac{\alpha}{\omega_1}Q_2 + a_3b_3 - z \geq 0$

$\Rightarrow \frac{-Q_2-\sqrt{Q_2^2-4P_2(a_3b_3-z)}}{2P_2} \geq \frac{\alpha}{\omega_1} \geq \frac{-Q_2+\sqrt{Q_2^2-4P_2(a_3b_3-z)}}{2P_2}$

$\Rightarrow \omega_1 \frac{-Q_2-\sqrt{Q_2^2-4P_2(a_3b_3-z)}}{2P_2} \geq \alpha \geq \omega_1 \frac{-Q_2+\sqrt{Q_2^2-4P_2(a_3b_3-z)}}{2P_2}$

$\Rightarrow \omega_1 \frac{Q_2+\sqrt{Q_2^2-4P_2(a_3b_3-z)}}{2P_2} \leq \alpha \leq \omega_1 \frac{Q_2-\sqrt{Q_2^2-4P_2(a_3b_3-z)}}{2P_2}$

Let, $\mu_C^R(z) = \omega_1 \frac{Q_2-\sqrt{Q_2^2-4P_2(a_3b_3-z)}}{2P_2}$

Now, $\frac{d}{dz}\mu_C^R(z) = \frac{d}{dz}\omega_1 \frac{Q_2-\sqrt{Q_2^2-4P_2(a_3b_3-z)}}{2P_2} = \frac{\omega_1}{2P_2}\frac{d}{dz}\left\{Q_2 - \sqrt{Q_2^2 - 4P_2(a_3b_3 - z)}\right\}$

$= \frac{\omega_1}{2P_2}\left\{-0 - \frac{1}{2}\left(Q_2^2 - 4P_2(a_3b_3 - z)\right)^{-\frac{1}{2}}\frac{d}{dz}\left(Q_2^2 - 4P_2(a_3b_3 - z)\right)\right\}$

$= \frac{\omega_1}{2P_1}\left\{-\frac{1}{2}\left(Q_2^2 - 4P_2(a_3b_3 - z)\right)^{-\frac{1}{2}}\left(0 - 4P_2(0 - 1)\right)\right\} = \frac{\omega_1}{4P_2}\left\{\frac{-4P_2}{\sqrt{\left(Q_2^2-4P_2(a_3b_3-z)\right)}}\right\}$

$= -\frac{\omega_1}{\sqrt{\left(Q_2^2-4P_2(a_3b_3-z)\right)}} = -\frac{\omega_1}{\sqrt{\left(\{a_3(b_3-b_2)+b_3(a_3-a_2)\}^2-4(a_3-a_2)(b_3-b_2)(a_3b_3-z)\right)}} < 0$

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Therefore, $\mu_C^R(z)$ is a decreasing function.

Also, $\mu_C^R(a_3b_3) = \omega_1 \frac{Q_2 - \sqrt{Q_2^2 - 4P_2(a_3b_3 - a_3b_3)}}{2P_2} = \omega_1 \frac{Q_2 - \sqrt{Q_2^2 - 0}}{2P_2} = \omega_1 \frac{Q_2 - Q_2}{2P_2} = 0$, $\mu_C^R(a_2b_2) = \omega_1$ and $\mu_C^R\left(\frac{a_2b_2 + a_3b_3}{2}\right) < \frac{\omega_1}{2}$

So the positive membership function of $C = A \times B$ is

$$\mu_C(z) = \begin{cases} \omega_1 \frac{-Q_1 + \sqrt{Q_1^2 - 4P_1(a_1b_1 - z)}}{2P_1} & ;\ a_1b_1 \le z \le a_2b_2 \\ \omega_1 & ;\ z = a_2b_2 \\ \omega_1 \frac{Q_2 - \sqrt{Q_2^2 - 4P_2(a_3b_3 - z)}}{2P_2} & ;\ a_2b_2 \le z \le a_3b_3 \\ 0 & ;\ otherwise \end{cases}$$

Similarly, we can find the neutral membership function.

$$\eta_C(z) = \begin{cases} \omega_2 \frac{-Q_1 + \sqrt{Q_1^2 - 4P_1(a_1b_1 - z)}}{2P_1} & ;\ a_1b_1 \le z \le a_2b_2 \\ \omega_2 & ;\ z = a_2b_2 \\ \omega_2 \frac{Q_2 - \sqrt{Q_2^2 - 4P_2(a_3b_3 - z)}}{2P_2} & ;\ a_2b_2 \le z \le a_3b_3 \\ 0 & ;\ otherwise \end{cases}$$

Now, for the negative membership function

$$C^\beta = [C_1(\beta), C_2(\beta)]$$
$$= [A_1(\beta), A_2(\beta)].[B_1(\beta), B_2(\beta)]$$
$$= [A_1(\beta)B_1(\beta), A_2(\beta)B_2(\beta)]$$
$$= \left[\frac{(a_2 - \omega_3 a_1) - \beta(a_2 - a_1)}{1 - \omega_3} \times \frac{(b_2 - \omega_3 b_1) - \beta(b_2 - b_1)}{1 - \omega_3}, \frac{\beta(a_3 - a_2) - (\omega_3 a_3 - a_2)}{1 - \omega_3} \times \frac{\beta(b_3 - b_2) - (\omega_3 b_3 - b_2)}{1 - \omega_3}\right]$$
$$= \left[\frac{\beta^2}{(1-\omega_3)^2}(a_2 - a_1)(b_2 - b_1) - \frac{\beta}{(1-\omega_3)}\{a_2(b_2 - b_1) + b_2(a_2 - a_1)\} + a_2b_2, \frac{\beta^2}{(1-\omega_3)^2}(a_3 - a_2)(b_3 - b_2) + \frac{\beta}{(1-\omega_3)}\{a_2(b_3 - b_2) + b_2(a_3 - a_2)\} + a_3b_3\right]$$

Let, $\frac{\beta^2}{(1-\omega_3)^2}(a_2 - a_1)(b_2 - b_1) - \frac{\beta}{(1-\omega_3)}\{a_2(b_2 - b_1) + b_2(a_2 - a_1)\} + a_2b_2 \le z \le \frac{\beta^2}{(1-\omega_3)^2}(a_3 - a_2)(b_3 - b_2) + \frac{\beta}{(1-\omega_3)}\{a_2(b_3 - b_2) + b_2(a_3 - a_2)\} + a_3b_3$

$$\frac{\beta^2}{(1-\omega_3)^2}(a_2 - a_1)(b_2 - b_1) - \frac{\beta}{(1-\omega_3)}\{a_2(b_2 - b_1) + b_2(a_2 - a_1)\} + a_2b_2 \le z$$

Let, $P_1' = (a_2 - a_1)(b_2 - b_1)$ and $Q_1' = a_2(b_2 - b_1) + b_2(a_2 - a_1)$

Now, $\frac{\beta^2}{(1-\omega_3)^2} P_1' - \frac{\beta}{(1-\omega_3)} Q_1' + a_2b_2 \le z$

$$\Rightarrow \frac{\beta^2}{(1-\omega_3)^2} P_1' - \frac{\beta}{(1-\omega_3)} Q_1' + a_2b_2 - z \le 0$$

$$\Rightarrow \frac{Q_1' - \sqrt{{Q_1'}^2 - 4P_1'(a_2b_2 - z)}}{2P_1'} \le \frac{\beta}{(1-\omega_3)} \le \frac{Q_1' + \sqrt{{Q_1'}^2 - 4P_1'(a_2b_2 - z)}}{2P_1'}$$

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$$\Rightarrow (1-\omega_3)\frac{Q_1'-\sqrt{Q_1'^2-4P_1'(a_2b_2-z)}}{2P_1'} \leq \beta \leq (1-\omega_3)\frac{Q_1'+\sqrt{Q_1'^2-4P_1'(a_2b_2-z)}}{2P_1'}$$

Let, $\nu_C^L(z) = (1-\omega_3)\frac{Q_1'-\sqrt{Q_1'^2-4P_1'(a_2b_2-z)}}{2P_1'}$

Now, $\frac{d}{dz}\nu_C^L(z) = \frac{d}{dz}(1-\omega_3)\frac{Q_1'-\sqrt{Q_1'^2-4P_1'(a_2b_2-z)}}{2P_1'}$

$$= -\frac{(1-\omega_3)}{\sqrt{(\{a_2(b_2-b_1)+b_2(a_2-a_1)\}^2-4(a_2-a_1)(b_2-b_1)(a_2b_2-z))}} < 0$$

Therefore, $\nu_C^L(z)$ is a decreasing function.

Also, $\nu_C^L(a_1b_1) = 1$, $\nu_C^L(a_2b_2) = 0$ and $\nu_C^L\left(\frac{a_2b_2+a_1b_1}{2}\right) < \frac{(1+\omega_3)}{2}$

Again,

$$\frac{\beta^2}{(1-\omega_3)^2}(a_3-a_2)(b_3-b_2) + \frac{\beta}{(1-\omega_3)}\{a_2(b_3-b_2)+b_2(a_3-a_2)\} + a_3b_3 \geq z$$

Let, $P_2' = (a_3-a_2)(b_3-b_2)$ and $Q_2' = a_2(b_3-b_2)+b_2(a_3-a_2)$, then

Now, $\frac{\beta^2}{(1-\omega_3)^2}P_2' + \frac{\beta}{(1-\omega_3)}Q_2' + a_3b_3 \geq z$

$$\Rightarrow \frac{\beta^2}{(1-\omega_3)^2}P_2' + \frac{\beta}{(1-\omega_3)}Q_2' + a_3b_3 - z \geq 0$$

$$\Rightarrow \frac{-Q_2'-\sqrt{Q_2'^2-4P_2'(a_3b_3-z)}}{2P_2'} \geq \frac{\beta}{(1-\omega_3)} \geq \frac{-Q_2'+\sqrt{Q_2'^2-4P_2'(a_3b_3-z)}}{2P_2'}$$

$$\Rightarrow (1-\omega_3)\frac{-Q_2'-\sqrt{Q_2'^2-4P_2'(a_3b_3-z)}}{2P_2'} \geq \beta \geq (1-\omega_3)\frac{-Q_2'+\sqrt{Q_2'^2-4P_2'(a_3b_3-z)}}{2P_2'}$$

Let, $\nu_C^R(z) = (1-\omega_3)\frac{-Q_2'+\sqrt{Q_2'^2-4P_2'(a_3b_3-z)}}{2P_2'}$

Now, $\frac{d}{dz}\nu_C^R(z) = \frac{d}{dz}(1-\omega_3)\frac{-Q_2'+\sqrt{Q_2'^2-4P_2'(a_3b_3-z)}}{2P_2'}$

$$= \frac{(1-\omega_3)}{\sqrt{(\{a_2(b_3-b_2)+b_2(a_3-a_2)\}^2-4(a_3-a_2)(b_3-b_2)(a_3b_3-z))}} > 0$$

Therefore, $\nu_C^R(z)$ is an increasing function.

Also, $\nu_C^R(a_2b_2) = \omega_3$, $\nu_C^R(a_3b_3) = 1$ and $\nu_C^R\left(\frac{a_2b_2+a_3b_3}{2}\right) > \frac{(1+\omega_3)}{2}$

So the negative membership function of $C = A \times B$ is

$$\nu_C(z) = \begin{cases} (1-\omega_3)\frac{Q_1'-\sqrt{Q_1'^2-4P_1'(a_2b_2-z)}}{2P_1'} & ;\ a_1b_1 \leq z \leq a_2b_2 \\ \omega_3 & ;\ z = a_2b_2 \\ (1-\omega_3)\frac{-Q_2'+\sqrt{Q_2'^2-4P_2'(a_3b_3-z)}}{2P_2'} & ;\ a_2b_2 \leq z \leq a_3b_3 \\ 1 & ;\ otherwise \end{cases}$$

4. Illustrations

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4.1. Numerical Example 1: Let $A = \langle(2,3,4,5); 0.6, 0.2, 0.1\rangle$ be a positive trapezoidal picture fuzzy number where the positive , neutral and negative membership functions are as follows:

$$\mu_A(x) = \begin{cases} 0 & ; x < 2 \\ 0.6(x-2) & ; 2 \le x < 3 \\ 0.6 & ; 3 \le x \le 4 \\ 0.6(5-x) & ; 4 < x \le 5 \\ 0 & ; x > 5 \end{cases}, \quad \eta_A(x) = \begin{cases} 0 & ; x < 2 \\ 0.2(x-2) & ; 2 \le x < 3 \\ 0.2 & ; 3 \le x \le 4 \\ 0.2(5-x) & ; 4 < x \le 5 \\ 0 & ; x > 5 \end{cases}$$

$$\nu_A(x) = \begin{cases} 1 & ; x < 2 \\ 0.9(2-x)+1 & ; 2 \le x < 3 \\ 0.1 & ; 3 \le x \le 4 \\ 0.9(x-4)+0.1 & ; 4 < x \le 5 \\ 1 & ; x > 5 \end{cases}$$

The following **Fig. 3** is the graphical representations of the GTraPFNs $\boldsymbol{A}$:

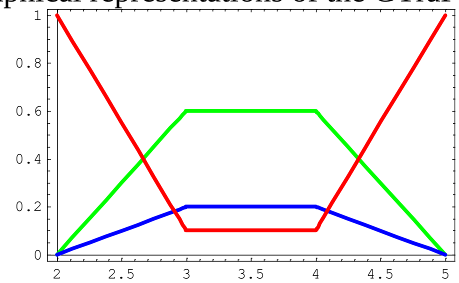


Fig. 3: GTraPFN $\boldsymbol{A}$

The corresponding (α, γ, β) −cut of the above trapezoidal picture fuzzy numbers A and B are as follows:

$$A^{(\alpha,\gamma,\beta)} = \left\{\left[2 + \frac{\alpha}{0.6}, 5 - \frac{\alpha}{0.6}\right], \left[2 + \frac{\gamma}{0.2}, 5 - \frac{\gamma}{0.2}\right], \left[2 - \frac{\beta-1}{0.9}, 4 + \frac{\beta-0.1}{0.9}\right]\right\}$$

Therefore,

$$18A^{(\alpha,\gamma,\beta)} = \{[36 + 30\alpha, 90 - 30\alpha], [36 + 90\gamma, 90 - 90\gamma], [36 - 20(\beta - 1), 72 + 20(\beta - 0.1)]\}$$

The corresponding positive, neutral and negative membership functions are as follows:

$$\mu_{18A}(x) = \begin{cases} 0 & ; x < 36 \\ \frac{x-36}{30} & ; 36 \le x < 54 \\ 0.6 & ; 54 \le x \le 72 \\ \frac{90-x}{30} & ; 72 < x \le 90 \\ 0 & ; x > 90 \end{cases}, \quad \eta_{18A}(x) = \begin{cases} 0 & ; x < 36 \\ \frac{x-36}{90} & ; 36 \le x < 54 \\ 0.2 & ; 54 \le x \le 72 \\ \frac{90-x}{90} & ; 72 < x \le 90 \\ 0 & ; x > 90 \end{cases}, \quad \nu_{18A}(x) =$$

$$\begin{cases} 1 & ; x < 36 \\ \frac{36-x}{20} + 1 & ; 36 \le x < 54 \\ 0.1 & ; 54 \le x \le 72 \\ \frac{x-72}{20} + 0.1 & ; 72 < x \le 90 \\ 1 & ; x > 90 \end{cases}$$

The following **Fig. 4** is the graphical representations of the GTraPFNs $\boldsymbol{18A}$:

https://doi.org/10.71058/jodac.v9i5014

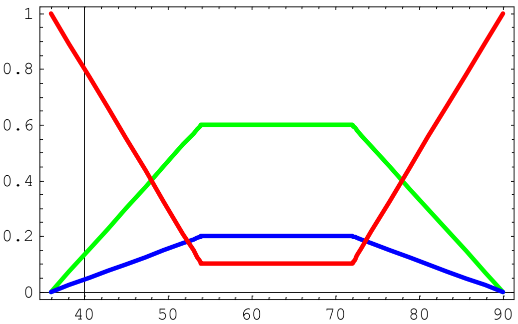


Fig. 4: GTraPFN 18*A*

The above figure implies that, the scalar multiplication of a generalized trapezoidal picture fuzzy number is also a generalized trapezoidal picture fuzzy number.

4.2. Numerical Example 2: Let $A = \langle(2,3,4,5); 0.6, 0.2, 0.1\rangle$ and $B = \langle(1,3,5,7); 0.6, 0.2, 0.1\rangle$ be two positive trapezoidal picture fuzzy numbers where the positive , neutral and negative membership functions are as follows:

$$\mu_A(x) = \begin{cases} 0 & ; x < 2 \\ 0.6(x-2) & ; 2 \le x < 3 \\ 0.6 & ; 3 \le x \le 4 \\ 0.6(5-x) & ; 4 < x \le 5 \\ 0 & ; x > 5 \end{cases}, \quad \eta_A(x) = \begin{cases} 0 & ; x < 2 \\ 0.2(x-2) & ; 2 \le x < 3 \\ 0.2 & ; 3 \le x \le 4 \\ 0.2(5-x) & ; 4 < x \le 5 \\ 0 & ; x > 5 \end{cases}$$

$$\nu_A(x) = \begin{cases} 1 & ; x < 2 \\ 0.9(2-x)+1 & ; 2 \le x < 3 \\ 0.1 & ; 3 \le x \le 4 \\ 0.9(x-4)+0.1 & ; 4 < x \le 5 \\ 1 & ; x > 5 \end{cases} \text{ and } \mu_B(x) = \begin{cases} 0 & ; x < 1 \\ \frac{0.6(x-1)}{2} & ; 1 \le x < 3 \\ 0.6 & ; 3 \le x \le 5 \\ \frac{0.6(7-x)}{2} & ; 5 < x \le 7 \\ 0 & ; x > 7 \end{cases},$$

$$\eta_B(x) = \begin{cases} 0 & ; x < 1 \\ \frac{0.2(x-1)}{2} & ; 1 \le x < 3 \\ 0.2 & ; 3 \le x \le 5 \\ \frac{0.2(7-x)}{2} & ; 5 < x \le 7 \\ 0 & ; x > 7 \end{cases}, \quad \nu_B(x) = \begin{cases} 1 & ; x < 1 \\ 0.45(1-x)+1 & ; 1 \le x < 3 \\ 0.1 & ; 3 \le x \le 5 \\ 0.45(x-7)+1 & ; 5 < x \le 7 \\ 1 & ; x > 7 \end{cases}$$

The following **Fig. 5** and **Fig. 6** are the graphical representations of the GTraPFNs ***A*** and ***B***:

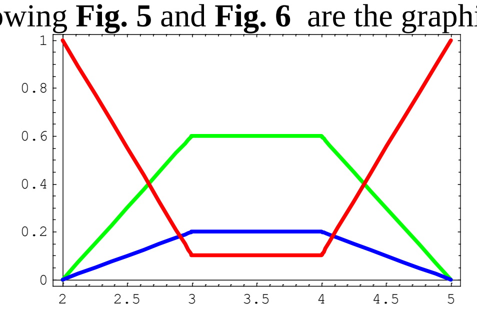


Fig. 5: GTraPFN *A*

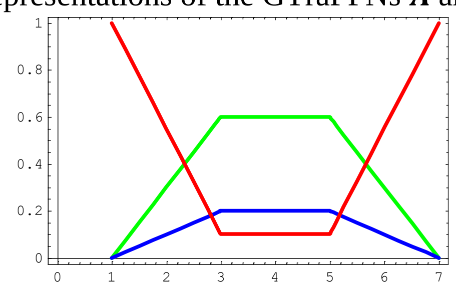


Fig. 6: GTraPFN *B*

The corresponding (α, γ, β) −cut of the above trapezoidal picture fuzzy numbers A and B are as follows:

$$A^{(\alpha,\gamma,\beta)} = \left\{\left[2 + \frac{\alpha}{0.6}, 5 - \frac{\alpha}{0.6}\right], \left[2 + \frac{\gamma}{0.2}, 5 - \frac{\gamma}{0.2}\right], \left[2 - \frac{\beta-1}{0.9}, 4 + \frac{\beta-0.1}{0.9}\right]\right\}$$

https://doi.org/10.71058/jodac.v9i5014

$$B^{(\alpha,\gamma,\beta)} = \left\{\left[1+\frac{\alpha}{0.3}, 7-\frac{\alpha}{0.3}\right], \left[1+\frac{\gamma}{0.1}, 7-\frac{\gamma}{0.1}\right], \left[1-\frac{\beta-1}{0.45}, 7+\frac{\beta-1}{0.45}\right]\right\}$$

Now,

$$(A \div B)^{(\alpha,\gamma,\beta)} = \left\{\left[\left(2+\frac{\alpha}{0.6}\right)/\left(7-\frac{\alpha}{0.3}\right), \left(5-\frac{\alpha}{0.6}\right)/\left(1+\frac{\alpha}{0.3}\right)\right], \left[\left(2+\frac{\gamma}{0.2}\right)/\left(7-\frac{\gamma}{0.1}\right), \left(5-\frac{\gamma}{0.2}\right)/\left(1+\frac{\gamma}{0.1}\right)\right], \left[\left(2-\frac{\beta-1}{0.9}\right)/\left(7+\frac{\beta-1}{0.45}\right), \left(4+\frac{\beta-0.1}{0.9}\right)/\left(1-\frac{\beta-1}{0.45}\right)\right]\right\}$$

$$= \left\{\left[\frac{(6+5\alpha)}{(21-10\alpha)}, \frac{(15-5\alpha)}{(3+10\alpha)}\right], \left[\frac{(2+5\gamma)}{(7-10\gamma)}, \frac{(5-5\gamma)}{(1+10\gamma)}\right], \left[\frac{(28-10\beta)}{(43+20\beta)}, \frac{(35+10\beta)}{(29-20\beta)}\right]\right\}$$

$$(A \times B)^{(\alpha,\gamma,\beta)} = \left\{\left[\left(2+\frac{\alpha}{0.6}\right)\left(1+\frac{\alpha}{0.3}\right), \left(5-\frac{\alpha}{0.6}\right)\left(7-\frac{\alpha}{0.3}\right)\right], \left[\left(2+\frac{\gamma}{0.2}\right)\left(1+\frac{\gamma}{0.1}\right), \left(5-\frac{\gamma}{0.2}\right)\left(7-\frac{\gamma}{0.1}\right)\right], \left[\left(2-\frac{\beta-1}{0.9}\right)\left(1-\frac{\beta-1}{0.45}\right), \left(4+\frac{\beta-0.1}{0.9}\right)\left(7+\frac{\beta-1}{0.45}\right)\right]\right\}$$

$$= \left\{\left[\frac{50\alpha^2+75\alpha+18}{9}, \frac{50\alpha^2-255\alpha+315}{9}\right], [50\gamma^2+25\gamma+2, 50\gamma^2-85\gamma+35], \left[\frac{200\beta^2-850\beta+812}{81}, \frac{200\beta^2+1130\beta+1505}{81}\right]\right\}$$

Thus, the corresponding positive, neutral and negative membership functions are as follows:

$$\mu_{A\div B}(x) = \begin{cases} 0 & ; x \le \frac{2}{7} \\ \frac{21x-6}{10x+5} & ; \frac{2}{7} < x < \frac{3}{5} \\ 0.6 & ; \frac{3}{5} \le x \le \frac{4}{3} \\ \frac{15-3x}{5+10x} & ; \frac{4}{3} < x < 5 \\ 0 & ; x \ge 5 \end{cases}, \quad \eta_{A\div B}(x) = \begin{cases} 0 & ; x \le \frac{2}{7} \\ \frac{7x-2}{10x+5} & ; \frac{2}{7} < x < \frac{3}{5} \\ 0.2 & ; \frac{3}{5} \le x \le \frac{4}{3} \\ \frac{5-x}{5+10x} & ; \frac{4}{3} < x < 5 \\ 0 & ; x \ge 5 \end{cases}, \quad \nu_{A\div B}(x) = \begin{cases} 1 & ; x \le \frac{2}{7} \\ \frac{28-43x}{10+20x} & ; \frac{2}{7} < x < \frac{3}{5} \\ 0.1 & ; \frac{3}{5} \le x \le \frac{4}{3} \\ \frac{29x-35}{20x+10} & ; \frac{4}{3} < x < 5 \\ 1 & ; x \ge 5 \end{cases}$$

$$\mu_{A\times B}(x) = \begin{cases} 0 & ; x \le 2 \\ \frac{-15\pm\sqrt{81+72x}}{20} & ; 2 < x < 9 \\ 0.6 & ; 9 \le x \le 20 \\ \frac{51\pm\sqrt{81+72x}}{20} & ; 20 < x < 35 \\ 0 & ; x \ge 35 \end{cases}, \quad \eta_{A\times B}(x) = \begin{cases} 0 & ; x \le 2 \\ \frac{-5\pm\sqrt{9+8x}}{20} & ; 2 < x < 9 \\ 0.2 & ; 9 \le x \le 20 \\ \frac{17\pm\sqrt{9+8x}}{20} & ; 20 < x < 35 \\ 0 & ; x \ge 35 \end{cases}$$

$$\nu_{A\times B}(x) = \begin{cases} 1 & ; x \le 2 \\ \frac{170\pm\sqrt{2916+2592x}}{80} & ; 2 < x < 9 \\ 0.1 & ; 9 \le x \le 20 \\ \frac{-226\pm\sqrt{2916+2592x}}{80} & ; 20 < x < 35 \\ 1 & ; x \ge 35 \end{cases}$$

The following **Fig. 7** and **Fig. 8** are the graphical representations of the $\boldsymbol{A} \div \boldsymbol{B}$ and $\boldsymbol{A} \times \boldsymbol{B}$:

https://doi.org/10.71058/jodac.v9i5014

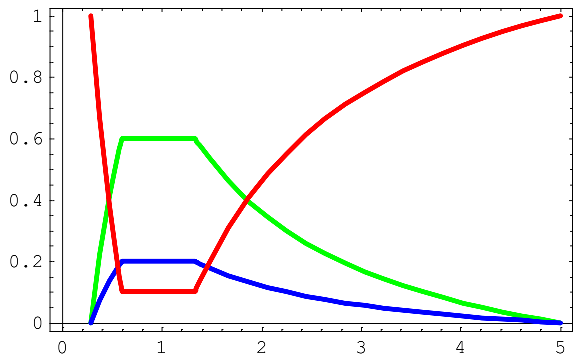


Fig. 7: $A \div B$

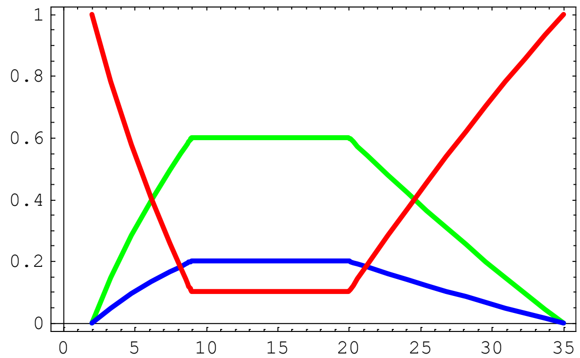


Fig. 8: $A \times B$

The above two figures imply that, the division and multiplication of two generalized trapezoidal picture fuzzy numbers are not generalized trapezoidal picture fuzzy numbers.

4.3. Numerical Example 3: Let $A = \langle(1,2,3); 0.5,0.3,0.2\rangle$ be a positive triangular picture fuzzy number where the positive , neutral and negative membership functions are as follows:

$$\mu_A(x) = \begin{cases} 0 & ; x < 1 \\ 0.5(x-1) & ; 1 \le x \le 2 \\ 0.5(3-x) & ; 2 < x \le 3 \\ 0 & ; x > 3 \end{cases}, \quad \eta_A(x) = \begin{cases} 0 & ; x < 1 \\ 0.3(x-1) & ; 1 \le x \le 2 \\ 0.3(3-x) & ; 2 < x \le 3 \\ 0 & ; x > 3 \end{cases},$$

$$\nu_A(x) = \begin{cases} 1 & ; x < 1 \\ 0.8(1-x)+1 & ; 1 \le x \le 2 \\ 0.8(x-3)+1 & ; 2 < x \le 3 \\ 1 & ; x > 3 \end{cases}$$

The corresponding (α, γ, β) −cut of the above triangular picture fuzzy number A is as follows:

$$A^{(\alpha,\gamma,\beta)} = \left\{\left[1 + \frac{\alpha}{0.5}, 3 - \frac{\alpha}{0.5}\right], \left[1 + \frac{\gamma}{0.3}, 3 - \frac{\gamma}{0.3}\right], \left[1 - \frac{\beta - 1}{0.8}, 3 + \frac{\beta - 1}{0.8}\right]\right\}$$

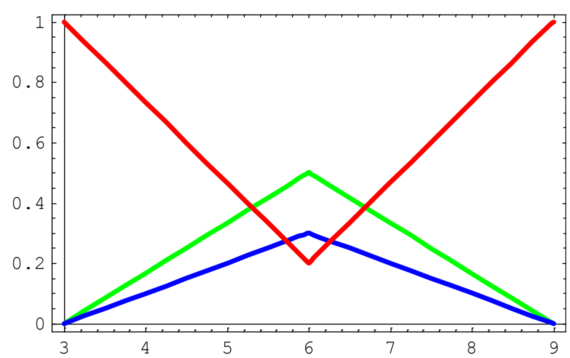
Therefore,

$$3A^{(\alpha,\gamma,\beta)} = \{[3 + 6\alpha, 9 - 6\alpha], [3 + 10\gamma, 9 - 10\gamma], [3 - 3.75(\beta - 1), 9 + 3.75(\beta - 1)]\}$$

The corresponding positive, neutral and negative membership functions are as follows:

$$\mu_{3A}(x) = \begin{cases} 0 & ; x < 3 \\ \frac{x-3}{6} & ; 3 \le x \le 6 \\ \frac{9-x}{6} & ; 6 < x \le 9 \\ 0 & ; x > 9 \end{cases}, \quad \eta_{3A}(x) = \begin{cases} 0 & ; x < 3 \\ \frac{x-3}{10} & ; 3 \le x \le 6 \\ \frac{9-x}{10} & ; 6 < x \le 9 \\ 0 & ; x > 9 \end{cases}, \quad \nu_{3A}(x) = \begin{cases} 1 & ; x < 3 \\ \frac{3-x}{3.75} + 1 & ; 3 \le x \le 6 \\ \frac{x-9}{3.75} + 1 & ; 6 < x \le 9 \\ 1 & ; x > 9 \end{cases}$$

The following **Fig. 9** is the graphical representations of the **$3A$**:



https://doi.org/10.71058/jodac.v9i5014

Fig. 9: $3A$

The above figure implies that, the scalar multiplication of a generalized triangular picture fuzzy number is a generalized triangular picture fuzzy number.

4.4. Numerical Example 4: Let $A = \langle (1,2,3); 0.5,0.3,0.2 \rangle$ and $B = \langle (2,3,4); 0.5,0.3,0.2 \rangle$ be two positive triangular picture fuzzy numbers where the positive , neutral and negative membership functions are as follows:

$$\mu_A(x) = \begin{cases} 0 & ; x < 1 \\ 0.5(x-1) & ; 1 \le x \le 2 \\ 0.5(3-x) & ; 2 < x \le 3 \\ 0 & ; x > 3 \end{cases}, \quad \eta_A(x) = \begin{cases} 0 & ; x < 1 \\ 0.3(x-1) & ; 1 \le x \le 2 \\ 0.3(3-x) & ; 2 < x \le 3 \\ 0 & ; x > 3 \end{cases},$$

$$\nu_A(x) = \begin{cases} 1 & ; x < 1 \\ 0.8(1-x)+1 & ; 1 \le x \le 2 \\ 0.8(x-3)+1 & ; 2 < x \le 3 \\ 1 & ; x > 3 \end{cases} \text{ and } \mu_B(x) = \begin{cases} 0 & ; x < 2 \\ 0.5(x-2) & ; 2 \le x \le 3 \\ 0.5(4-x) & ; 3 < x \le 4 \\ 0 & ; x > 4 \end{cases},$$

$$\eta_B(x) = \begin{cases} 0 & ; x < 2 \\ 0.3(x-2) & ; 2 \le x \le 3 \\ 0.3(4-x) & ; 3 < x \le 4 \\ 0 & ; x > 4 \end{cases}, \quad \nu_B(x) = \begin{cases} 1 & ; x < 2 \\ 0.8(2-x)+1 & ; 2 \le x \le 3 \\ 0.8(x-4)+1 & ; 3 < x \le 4 \\ 1 & ; x > 4 \end{cases}$$

The following **Fig. 10** and **Fig. 11** are the graphical representations of the GTPFNs ***A*** and ***B***:

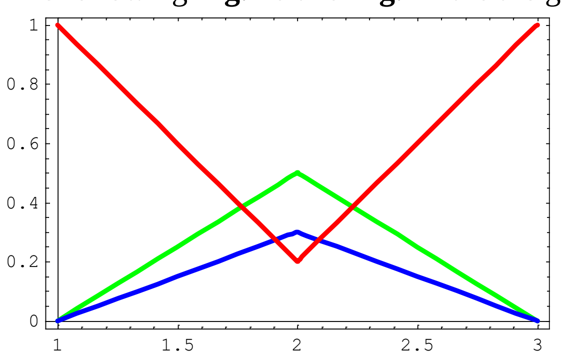


Fig. 10: GTPFN *A*

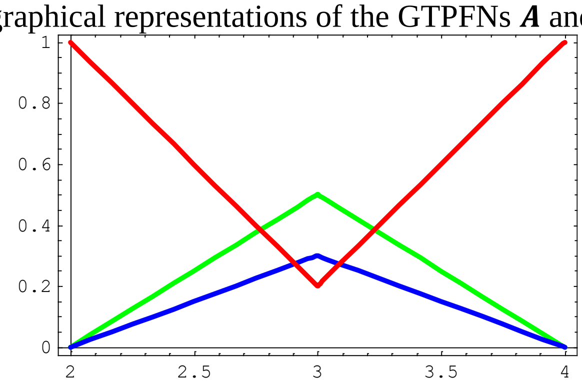


Fig. 11: GTPFN *B*

The corresponding (α, γ, β) −cut of the above triangular picture fuzzy numbers A and B are as follows:

$$A^{(\alpha,\gamma,\beta)} = \left\{ \left[1 + \frac{\alpha}{0.5}, 3 - \frac{\alpha}{0.5}\right], \left[1 + \frac{\gamma}{0.3}, 3 - \frac{\gamma}{0.3}\right], \left[1 - \frac{\beta - 1}{0.8}, 3 + \frac{\beta - 1}{0.8}\right] \right\}$$

$$B^{(\alpha,\gamma,\beta)} = \left\{ \left[2 + \frac{\alpha}{0.5}, 4 - \frac{\alpha}{0.5}\right], \left[2 + \frac{\gamma}{0.3}, 4 - \frac{\gamma}{0.3}\right], \left[2 - \frac{\beta - 1}{0.8}, 4 + \frac{\beta - 1}{0.8}\right] \right\}$$

Now,

$$(A \div B)^{(\alpha,\gamma,\beta)} = \left\{ \left[\left(1 + \frac{\alpha}{0.5}\right) / \left(4 - \frac{\alpha}{0.5}\right), \left(3 - \frac{\alpha}{0.5}\right) / \left(2 + \frac{\alpha}{0.5}\right) \right], \left[\left(1 + \frac{\gamma}{0.3}\right) / \left(4 - \frac{\gamma}{0.3}\right), \left(3 - \frac{\gamma}{0.3}\right) / \left(2 + \frac{\gamma}{0.3}\right) \right], \left[\left(1 - \frac{\beta-1}{0.8}\right) / \left(4 + \frac{\beta-1}{0.8}\right), \left(3 + \frac{\beta-1}{0.8}\right) / \left(2 - \frac{\beta-1}{0.8}\right) \right] \right\}$$

$$= \left\{ \left[\frac{0.5+\alpha}{2-\alpha}, \frac{1.5-\alpha}{1+\alpha} \right], \left[\frac{0.3+\gamma}{1.2-\gamma}, \frac{0.9-\gamma}{0.6+\gamma} \right], \left[\frac{1.8-\beta}{2.2+\beta}, \frac{1.4+\beta}{2.6-\beta} \right] \right\}$$

$$(A \times B)^{(\alpha,\gamma,\beta)} = \left\{ \left[\left(1 + \frac{\alpha}{0.5}\right)\left(2 + \frac{\alpha}{0.5}\right), \left(3 - \frac{\alpha}{0.5}\right)\left(4 - \frac{\alpha}{0.5}\right) \right], \left[\left(1 + \frac{\gamma}{0.3}\right)\left(2 + \frac{\gamma}{0.3}\right), \left(3 - \frac{\gamma}{0.3}\right)\left(4 - \frac{\gamma}{0.3}\right) \right], \left[\left(1 - \frac{\beta-1}{0.8}\right)\left(2 - \frac{\beta-1}{0.8}\right), \left(3 + \frac{\beta-1}{0.8}\right)\left(4 + \frac{\beta-1}{0.8}\right) \right] \right\}$$

https://doi.org/10.71058/jodac.v9i5014

$$= \left\{ \begin{matrix} [4\alpha^2 + 6\alpha + 2\,, 4\alpha^2 - 14\alpha + 12], \left[\frac{100\gamma^2+90\gamma+18}{9}, \frac{100\gamma^2-210\gamma+108}{9}\right], \\ \left[\frac{25\beta^2-110\beta+117}{16}, \frac{25\beta^2+90\beta+77}{16}\right] \end{matrix} \right\}$$

The corresponding positive, neutral and negative membership functions are as follows:

$$\mu_{A\div B}(x) = \begin{cases} 0 & ; x < \frac{1}{4} \\ \frac{2x-0.5}{1+x} & ; \frac{1}{4} \le x \le \frac{2}{3} \\ \frac{1.5-x}{1+x} & ; \frac{2}{3} < x \le \frac{3}{2} \\ 0 & ; x > \frac{3}{2} \end{cases}, \quad \eta_{A\div B}(x) = \begin{cases} 0 & ; x < \frac{1}{4} \\ \frac{1.2x-0.3}{1+x} & ; \frac{1}{4} \le x \le \frac{2}{3} \\ \frac{0.9-0.6x}{1+x} & ; \frac{2}{3} < x \le \frac{3}{2} \\ 0 & ; x > \frac{3}{2} \end{cases}, \quad \nu_{A\div B}(x) = \begin{cases} 1 & ; x < \frac{1}{4} \\ \frac{1.8-2.2x}{1+x} & ; \frac{1}{4} \le x \le \frac{2}{3} \\ \frac{2.6x-1.4}{1+x} & ; \frac{2}{3} < x \le \frac{3}{2} \\ 1 & ; x > \frac{3}{2} \end{cases}$$

$$\mu_{A\times B}(x) = \begin{cases} 0 & ; x < 2 \\ \frac{-3\pm\sqrt{1+4x}}{4} & ; 2 \le x \le 6 \\ \frac{7\pm\sqrt{1+4x}}{4} & ; 6 < x \le 12 \\ 0 & ; x > 12 \end{cases}, \quad \eta_{A\times B}(x) = \begin{cases} 0 & ; x < 2 \\ \frac{-9\pm\sqrt{9+36x}}{20} & ; 2 \le x \le 6 \\ \frac{21\pm\sqrt{9+36x}}{20} & ; 6 < x \le 12 \\ 0 & ; x > 12 \end{cases},$$

$$\nu_{A\times B}(x) = \begin{cases} 1 & ; x < 2 \\ \frac{11\pm\sqrt{4+16x}}{5} & ; 2 \le x \le 6 \\ \frac{-9\pm\sqrt{4+16x}}{5} & ; 6 < x \le 12 \\ 1 & ; x > 12 \end{cases}$$

The following **Fig. 12** and **Fig. 13** are the graphical representations of the $\boldsymbol{A \div B}$ and $\boldsymbol{A \times B}$:

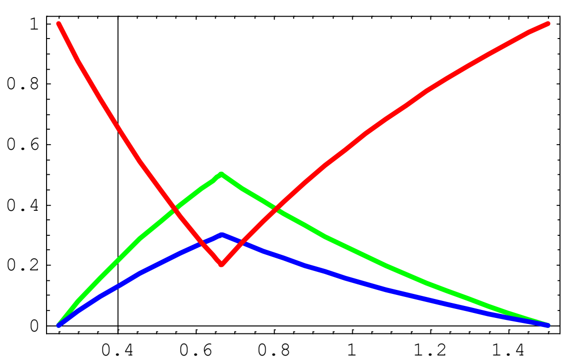


Fig. 12: $\boldsymbol{A \div B}$

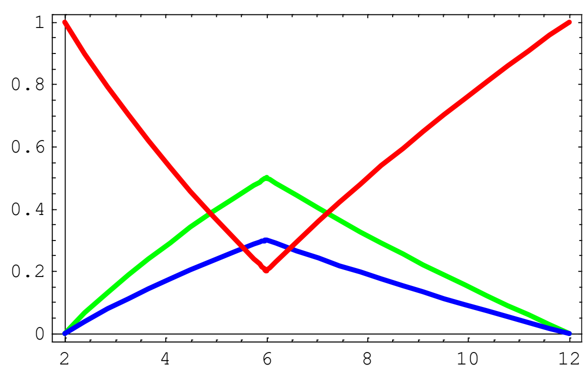


Fig. 13: $\boldsymbol{A \times B}$

The above two figures imply that, the division and multiplication of two generalized triangular picture fuzzy numbers are not generalized triangular picture fuzzy numbers.

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I am grateful to all my well-wishers for their continuous inspiration to pursuing my research work.

5. Conclusions

In our daily life, we are to meet many things where most of them are vague than precise and those thinks always cannot be described by the concept of the conventional classical set theory. The concepts of cut sets and picture fuzzy numbers are two important tools for describing fuzziness and the arithmetic operations. In this article, the scalar multiplication of a generalized trapezoidal and triangular picture fuzzy number and the division and multiplication of two generalized trapezoidal and triangular picture fuzzy numbers by $(\alpha, \gamma, \beta) -$ cut method are discussed with numerical examples to illustrate.

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