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Abstract: *The Fuzzy graphs serve as a practical tool in mathematics, enabling users to effortlessly represent relationships between various concepts. Their inherent fuzziness makes them adaptable and well- suited for diverse environments. A topological index is a numerical value that characterizes the structural graph of a molecule. This study aims to introduce a new topological index in the fuzzy graph theory and study its properties. In this direction, we have introduced* F^**-index of a fuzzy graph* G*. In this paper, the* F^**-index of fuzzy generalized transformation graphs and its related bounds are studied.*

Keywords: *First Zagreb index; Second Zagreb index; F-index; F*-index.*

1. Introduction

Graphs are commonly used to represent relationships between structures and objects within a given context, based on the available information. They serve as an effective tool for illustrating structures, offering insights that help in analyzing and understanding the behavior of the concepts being examined. A graph is composed of two fundamental components: a set of vertices and a set of edges. Vertices represent stationary points or objects in the environment, while edges denote the connections or relationships between these vertices. However, when there is uncertainty in determining the nature of the vertices or edges, it becomes necessary to describe the graph within the framework of fuzzy graphs, which account for such ambiguities.

In a graph G, the vertex set is non-empty and is denoted by V. The collection of edges E is defined by a symmetric binary relation on V. Similarly, a symmetric binary relation on a fuzzy subset leads to the development of the fuzzy graph model. The concept of fuzzy sets and fuzzy relations was introduced by Zadeh [12] , highlighting the significance of values that lie between the binary digits 0 and 1 (i.e., True or False). This idea paved the way for exploring the nature of uncertainty, which can also be described as vagueness, ambiguity, or similar terms. The term "fuzzy" was first coined by Zadeh in 1965 in his seminal paper. Further advancements were made by Rosenfeld [11], who explored fuzzy relations on fuzzy sets and introduced a novel approach to fuzzy graphs by combining graph theory with fuzziness. He explored various graph-related concepts such as fuzzy trees, fuzzy cycles, fuzzy bridges, and other properties within the framework of fuzzy graphs.

Topological indices (TIs) are numerical quantities of a graph that describe its topology and are calculated on the molecular graph of a chemical compound. In molecular graph vertex represents an atom and an edge represents the bond between two atoms. Zagreb indices are the degree-based TI's introduced by Gutman and Trinajstic [2], these were used to calculate π-electron energy of a conjugate system. In 2015, another degree-based topological index was introduced by Fortula and Gutman [3] named as 'Forgotten topological index'(F-index).

Islam and Pal have developed the concept of first Zagreb index [4] and F-index [5, 6] in fuzzy graphs(FGs), the first entire Zagreb index for fuzzy graphs studied in [8] For more study on topological indices of FGs one can refer [9, 7].

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2. Preliminaries

Some useful definitions are given here, most of them are taken from [10].

For a universal set X, a pair $S = (X, \mu)$ is called a fuzzy set where μ is a called membership function of S whose domain is X and co-domain is [0,1]. A FG is a triplet, $G = (V, \sigma, \mu)$, where V is called vertex set of the fuzzy graph with vertex membership function $\sigma: V \to [0,1]$ and edge membership function $\mu: V \times V \to [0,1]$ satisfying $\mu(x,y) \le \min\{\sigma(x), \sigma(y)\}$. The edge set of G is defined as $E = \{(x,y) \in V \times V : \mu(x,y) > 0\}$. Note that, the edge (x,y) and (y,x) are considered as same and sometimes it called the edge xy or yx. Some times we have denoted $G = (V, E)$ as a fuzzy graph with vertex set V and edge set E. Degree of a vertex $v \in V$ is defined as: $d(v) = \sum_{x \in V} \mu(xv)$. Let Δ and δ be the maximum and minimum degree of G, respectively. Suppose v_i and v_j are adjacent or nonadjacent vertices in G then it is denoted by $(v_i \sim v_j)$ and $(v_i \nsim v_j)$ respectively. Similarly, suppose v_i and e_j are incident or nonincident in G then it is denoted by $(v_i \sim e_j)$ and $(v_i \nsim e_j)$ respectively.

One of the key concepts in topological indices is the Zagreb index [2]. This index is particularly useful for understanding the complexity of parameters in chemical systems, biological systems, and other domains. Recently, this concept has been extended to fuzzy graph theory, which is defined independently many researchers which are explored through the following definitions:

Definition 1 [9]. Suppose $G = (V, \sigma, \mu)$ is a fuzzy graph. Then the first Zagreb index for fuzzy graph is defined as:

$$M(G) = \sum_{u \in V(G)} \sigma(u) d(u)^2$$

Definition 2 [4]. Suppose $G = (V, \sigma, \mu)$ is a fuzzy graph.Then the first Zagreb index for fuzzy graph is defined as:

$$M_1(G) = \sum_{u \in V(G)} [\sigma(u) d(u)]^2$$

Definition 3 [9]. Suppose $G = (V, \sigma, \mu)$ is a fuzzy graph. Then the second Zagreb index for fuzzy graph is defined as:

$$M_2(G) = \sum_{(u,v) \in E(G)} \sigma(u)\sigma(v) d(u) d(v)$$

Definition 4 [6]. Suppose $G = (V, \sigma, \mu)$ is a fuzzy graph. Then the F-index for fuzzy graph is defined as:

$$FI(G) = \sum_{u \in V(G)} [\sigma(u) d(u)]^3$$

3. Fuzzy Generalized Transformation Graphs

Let $G = (V, \sigma, \mu)$ be a fuzzy graph with underlying crisp graph H. Then fuzzy generalized transformation graph $F(G^{ab}) = (X, \sigma^{ab}, \mu^{ab})$, where $X = V(G) \cup E(G)$. Let α and β be any two elements of $V(G) \cup E(G)$. The associativity of α and β is (+) if they are adjacent or incident in G otherwise (-). Let ab be 2-permutations of the set$(+,-)$. We say that α and β corresponds to the first term a of ab if both α and β are in $V(G)$, where as α and β corresponds to the second term b of ab if one of α and β is in $V(G)$ and the other is in $E(G)$. For generalized transformation graphs of a crisp graph see [1].

Easily the four fuzzy generalized transformation graphs are obtained $G^{++}, G^{+-}, G^{-+}, G^{--}$. Membership values of every vertex and edge of these fuzzy generalized transformation graphs are given as below.

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1. Let G^{++} be fuzzy generalized transformation graph of a fuzzy graph $G = (V, \sigma, \mu)$, where $V(G^{++}) = V(G) \cup E(G)$. The membership value of every vertex and edge of G^{++} is defined as

$$\sigma_{G^{++}}(u) = \begin{cases} \sigma(u) \ \ if \ u \in V(G) \\ \mu(u) \ if \ u \in E(G) \end{cases}$$

$$\mu_{G^{++}}(v_i, v_j) = \begin{cases} \mu(v_i, v_j) \ \ if \ (v_i \sim v_j) \in G \\ \wedge [\sigma(v_i), \mu(e_j)] \ \ if \ if \ (v_i \sim e_j) \in G \end{cases}$$

2. Let G^{+-} be fuzzy generalized transformation graph of a fuzzy graph $G = (V, \sigma, \mu)$, where $V(G^{+-}) = V(G) \cup E(G)$. The membership value of every vertex and edge of G^{+-} is defined as

$$\sigma_{G^{+-}}(u) = \begin{cases} \sigma(u) \ \ if \ u \in V(G) \\ \mu(u) \ if \ u \in E(G) \end{cases}$$

$$\mu_{G^{+-}}(v_i, v_j) = \begin{cases} \mu(v_i, v_j) \ \ if \ (v_i \sim v_j) \in G \\ \wedge [\sigma(v_i), \mu(e_j)] \ \ if \ if \ (v_i \nsim e_j) \in G \end{cases}$$

3. Let G^{-+} be fuzzy generalized transformation graph of a fuzzy graph $G = (V, \sigma, \mu)$, where $V(G^{-+}) = V(G) \cup E(G)$. The membership value of every vertex and edge of G^{-+} is defined as

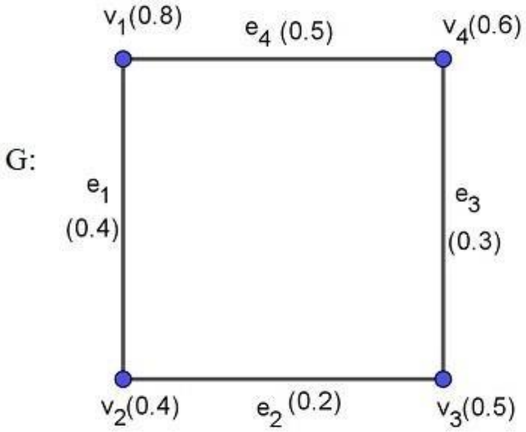
$$\sigma_{G^{-+}}(u) = \begin{cases} \sigma(u) \ \ if \ u \in V(G) \\ \mu(u) \ if \ u \in E(G) \end{cases}$$

$$\mu_{G^{-+}}(v_i, v_j) = \begin{cases} \wedge [\sigma(v_i), \sigma(v_j)] \ \ if \ (v_i \sim v_j) \in G \ and \\ \quad v_i \ and \ v_j \ are \ not \ adjacent \ in \ G \\ \wedge [\sigma(v_i), \mu(e_j)] \ \ if \ if \ (v_i \sim e_j) \in G \end{cases}$$

4. Let G^{--} be fuzzy generalized transformation graph of a fuzzy graph $G = (V, \sigma, \mu)$, where $V(G^{--}) = V(G) \cup E(G)$.The membership value of every vertex and edge of G^{--} is defined as

$$\sigma_{G^{--}}(u) = \begin{cases} \sigma(u) \ \ if \ u \in V(G) \\ \mu(u) \ if \ u \in E(G) \end{cases}$$

$$\mu_{G^{--}}(v_i, v_j) = \begin{cases} \wedge [\sigma(v_i), \sigma(v_j)] \ if \ (v_i \nsim v_j) \in G \\ \wedge [\sigma(v_i), \mu(e_j)] \ \ if \ if \ (v_i \nsim e_j) \in G \end{cases}$$



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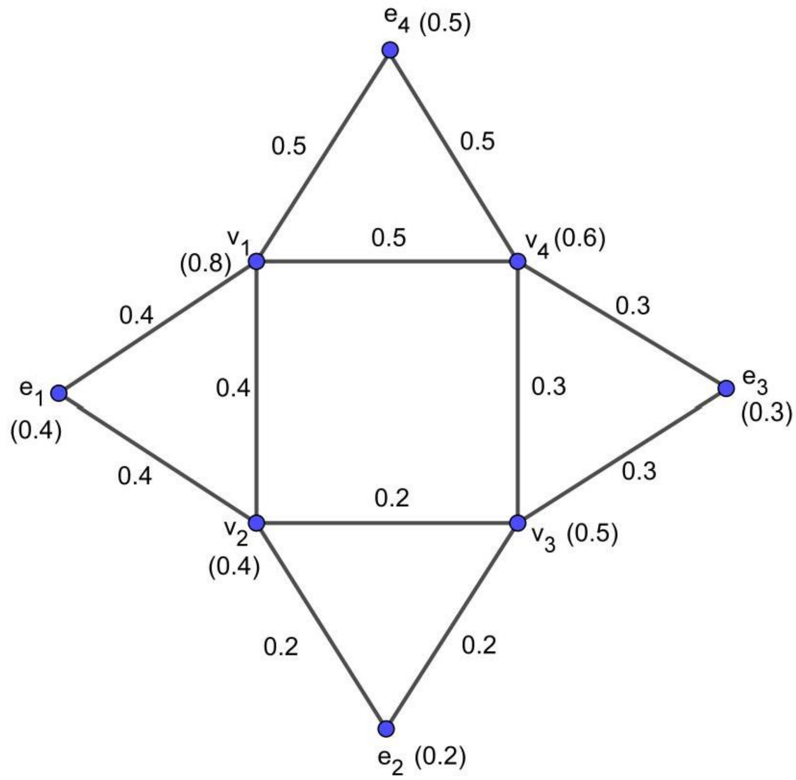
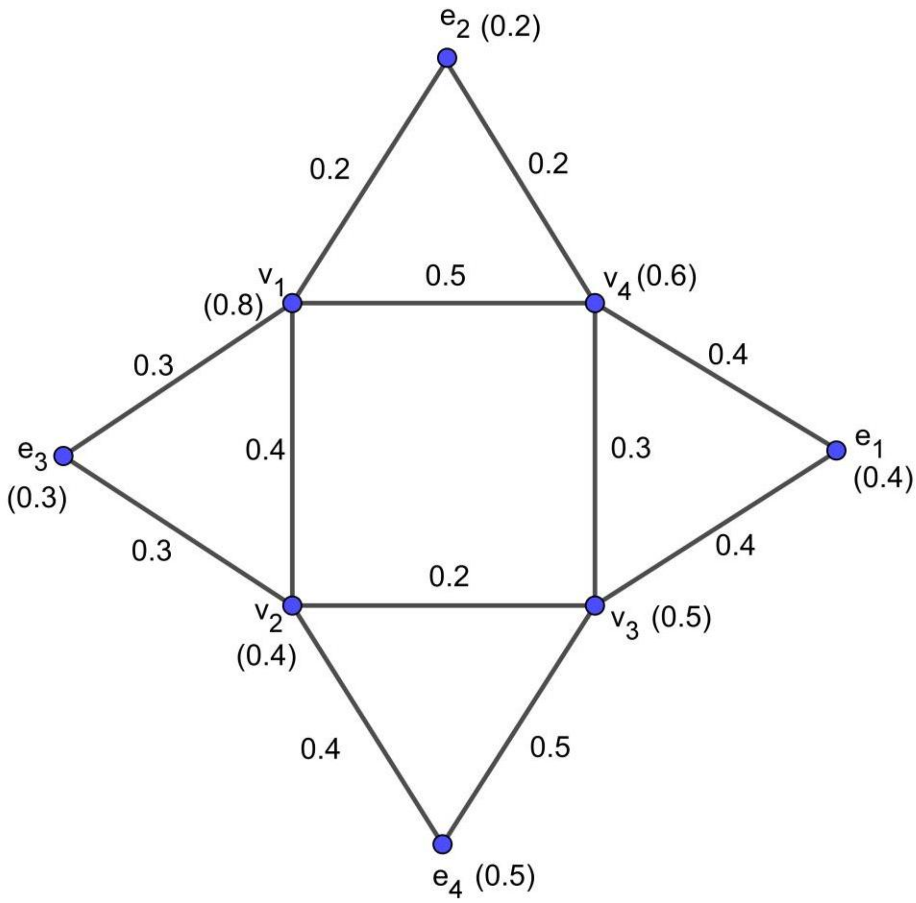
G^{++} :



G^{+-} :



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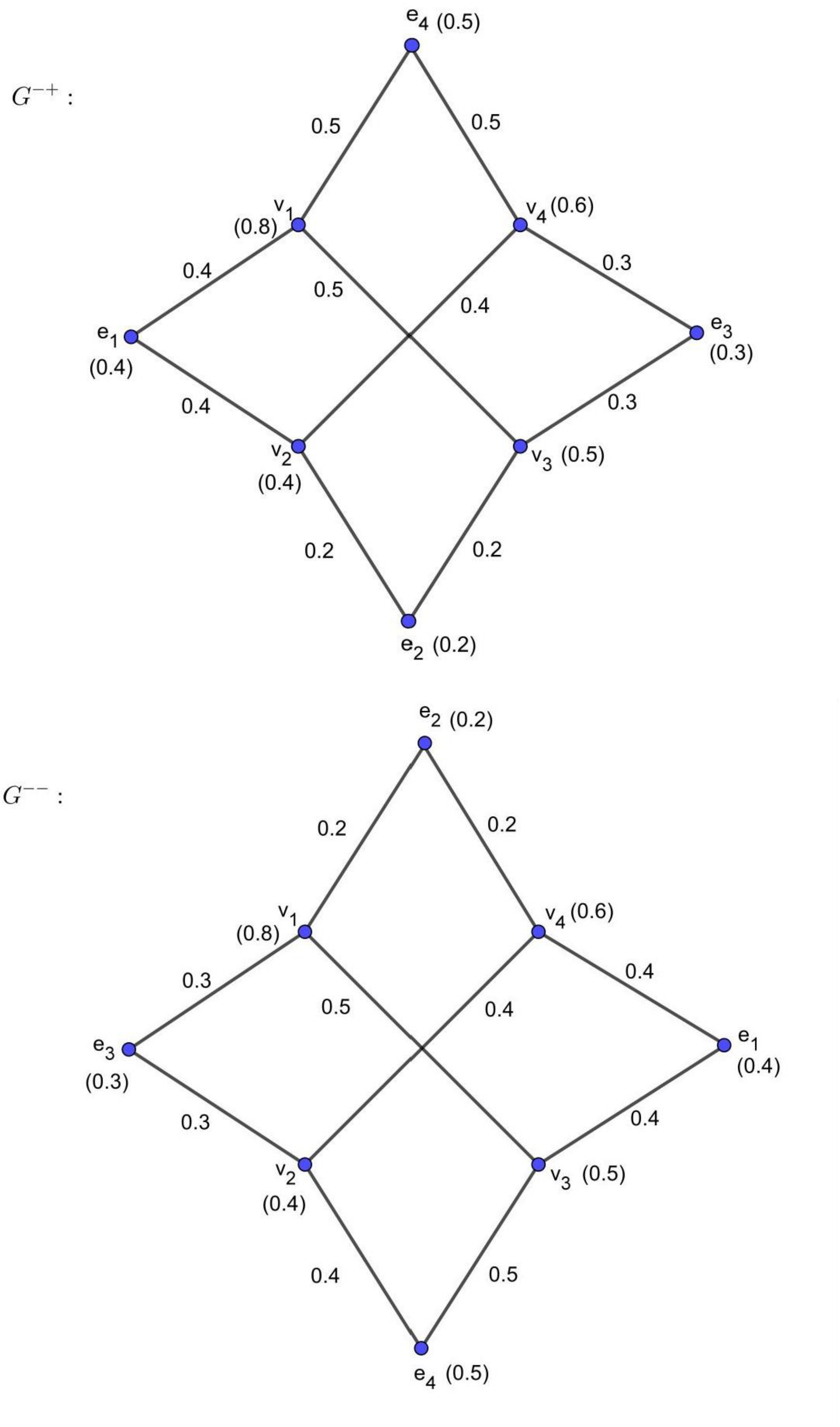

Figure 1. A fuzzy graph G and its fuzzy generalized transformation graphs

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A fuzzy graph G and its fuzzy generalized transformation graphs are shown in figure 1.

Note 1. Let H be a crisp graph and G be its fuzzy graph, similarly, H^{xy} be a generalized transformation crisp graph and G^{xy} be fuzzy generalized transformation graph. Then the number of vertices and edges of each graph is considered as follows:

Graph	Number of vertices	Number of edges
H	n	m
G	n_1	m_1
H^{xy}	n'	m'
G^{xy}	$n_1{}'$	$m_1{}'$

Motivated by the fuzzy topological indices like, first Zagreb index and the forgotten index of fuzzy graphs, here we define a new topological index in fuzzy graph theory namely the F^* −index of a fuzzy graph as follows:

Definition 5 *Let* $G = (V, \sigma, \mu)$ *be a fuzzy graph with underlying crisp graph* H*. Then* F^* *index of fuzzy graph* G *is defined as*

$$F^*(G) = \frac{2m}{n}\sum_{i=1}^{n_1}[\sigma(v_i)d_G(v_i)]^3 \tag{1}$$

Example

Consider a fuzzy graph G and its fuzzy generalized transformation graphs G^{++}, G^{+-}, G^{-+} and G^{--} are shown in figure 1. To compute $F^*(G)$, we need the following information:

Table 1. The membership values and degree of G.

Vertices	Membership values	Degree
v_1	0.8	0.9
v_2	0.4	0.6
v_3	0.5	0.5
v_4	0.6	0.8

Table 2. The membership values and degree of G^{++}

Vertices	Membership values	Degree
v_1	0.8	1.8
v_2	0.4	1.2
v_3	0.5	1.0
v_4	0.6	1.6
e_1	0.4	0.8
e_2	0.2	0.4
e_3	0.3	0.6
e_4	0.5	1.0

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Table 3. The membership values and degree of G^{+-}

Vertices	Membership values	Degree
v_1	0.8	1.4
v_2	0.4	1.3
v_3	0.5	1.4
v_4	0.6	1.4
e_1	0.4	0.8
e_2	0.2	0.4
e_3	0.3	0.6
e_4	0.5	0.9

Table 4. The membership values and degree of G^{-+}

Vertices	Membership values	Degree
v_1	0.8	1.4
v_2	0.4	1.0
v_3	0.5	1.0
v_4	0.6	1.2
e_1	0.4	0.8
e_2	0.2	0.4
e_3	0.3	0.6
e_4	0.5	1.0

Table 5. The membership values and degree of G^{--}

Vertices	Membership values	Degree
v_1	0.8	1.0
v_2	0.4	1.1
v_3	0.5	1.4
v_4	0.6	1.0
e_1	0.4	0.8
e_2	0.2	0.4
e_3	0.3	0.6
e_4	0.5	0.9

Employing eqn.(1) to the fuzzy graphs shown in figure 1 and using the information in Table (1)-(5), we get:

$$F^*(G) = \frac{2m}{n}\sum_{i=1}^{n_1}[\sigma(v_i)d_G(v_i)]^3 = 1.0262.$$

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$$F^*(G^{++}) = \frac{2m\prime}{n\prime}\sum_{i=1}^{n\prime_1} [\sigma(v_i)d_{G^{++}}(v_i)]^3 = 12.8103$$

$$F^*(G^{+-}) = \frac{2m\prime}{n\prime}\sum_{i=1}^{n\prime_1} [\sigma(v_i)d_{G^{+-}}(v_i)]^3 = 7.8339$$

$$F^*(G^{-+}) = \frac{2m\prime}{n\prime}\sum_{i=1}^{n\prime_1} [\sigma(v_i)d_{G^{-+}}(v_i)]^3 = 5.3277$$

$$F^*(G^{--}) = \frac{2m\prime}{n\prime}\sum_{i=1}^{n\prime_1} [\sigma(v_i)d_{G^{--}}(v_i)]^3 = 3.21567$$

4 Results

Lemma 1. Let $G = (V, \sigma, \mu)$ be a fuzzy graph and G^{ab} be fuzzy generalized transformation graph,then the degree of point vertex and line vertex is given by

1. $d_G^{++}(u) = 2d_G(u)$ *and* $d_G^{++}((e)) = 2\mu_G(e)$, where $e = (u, v) \in E(G)$
2. $d_G^{+-}(v_i) = d_G(v_i) + \sum_{v_i * e_j} \wedge [\sigma_G(v_i), \mu_G(e_j)]$ where $i \neq j$
   and $d_G^{+-}(e) = \sum_{v * e} \wedge [\mu_G(e), \sigma_G(v)]$, where $e = (u, v) \in E(G)$
3. $d_G^{-+}(v_i) = \sum_{v_i * v_j} \wedge [\sigma_G(v_i), \sigma_G(v_j)] + d_G(v_i)$
   and $d_G^{-+}(e) = 2\mu(e)$, where $e = (u, v) \in E(G)$
4. $d_G^{--}(u) = \sum_{v_i * v_j} \wedge [\sigma_G(v_i), \sigma_G(v_j)] + \sum_{v * e} \wedge [\mu_G(e), \sigma_G(v)]$
   and $d_G^{--}(e) = \sum_{v * e} \wedge [\mu_G(e), \sigma_G(v)]$.

Theorem 1 *Suppose $G = P_{n_1}$ is the fuzzy path graph and $H = P_n$ be the corresponding crisp graph. Then $F^*(P_{n_1}) \leqslant \frac{2m}{n}[8n_1 - 14]$*

Proof. Let $G = P(v_1, v_2, v_3 \cdots v_{n_1})$ be a fuzzy path with $V(G) = (v_1, v_2, v_3 \cdots v_{n_1})$. Then $d(v_1)=\mu_1$, $d(v_{n_1})=\mu_{n_1-1}$ and $d(v_i)=\mu_i + \mu_{i-1}$ for $i = 2,3 \cdots n_1 - 1$.

$$\therefore F^*(P_{n_1}) = \frac{2m}{n}\sum_{i=1}^{n_1} [\sigma(v_i)d(v_i)]^3$$
$$= \frac{2m}{n}[[\sigma(v_1)d(v_1)]^3 + [\ \sigma(v_{n_1})d(v_{n_1})]^3 + \sum_{i=2}^{n_1-1} [\sigma(v_i)d(v_i))]^3]$$
$$= \frac{2m}{n}[[\sigma(v_1)\mu_1]^3 + [\sigma(v_{n_1})(\mu_{n_1-1})]^3 + \sum_{i=2}^{n_1-1} [\sigma(v_i)(\mu_i + \mu_{i-1})]^3]$$

since $o \leq \sigma(v) \leq 1$ and $0 \leq \mu(u, v) \leq 1$.

$$\leq \frac{2m}{n}[1 + 1 + \sum_{i=2}^{n_1-1} [(1 + 1)]^3]$$
$$= \frac{2m}{n}[2 + \sum_{i=2}^{n_1-1} [2]^3]$$
$$= \frac{2m}{n}[2 + 8(n_1 - 2)]$$
$$= \frac{2m}{n}[8n_1 - 14]$$

Theorem 2 *Suppose $G = C_{n_1}$ is the fuzzy cycle and $H = C_n$ be the corresponding crisp graph. Then $F^*(C_{n_1}) \leqslant \frac{16m\,n_1}{n}$*

Proof. Let $G = C_{n_1}$ be a fuzzy cycle with $V(G) = (v_1, v_2, v_3 \dots v_{n_1})$. Then $d(v_1)=\mu_1 + \mu_{n_1}$ and $d(v_i)=\mu_i + \mu_{i-1}$ for $i = 2,3 \dots n_1$.

$$\therefore F^*(C_{n_1}) = \frac{2m}{n}\sum_{i=1}^{n_1} [\sigma(v_i)d(v_i)]^3$$
$$= \frac{2m}{n}[[\sigma(v_1)d(v_1)]^3 + \sum_{i=2}^{n_1} [\sigma(v_i)d(v_i))]^3]$$
$$= \frac{2m}{n}[[\sigma(v_1)(\mu_1 + \mu_{n_1})]^3 + \sum_{i=2}^{n_1} [\sigma(v_i)(\mu_i + \mu_{i-1})]^3]$$
$$\leq \frac{2m}{n}[(1 + 1)^3 + \sum_{i=2}^{n_1} [(1 + 1)]^3]$$

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$$= \frac{2m}{n}[2^3 + 2^3(n_1 - 1)]$$
$$= \frac{16mn_1}{n}.$$

Theorem 3 *Suppose* K_{1,n_1-1} *is fuzzy star graph and* $H = K_{1,n-1}$ *be the corresponding crisp graph. Then* $F^*(K_{1,n_1-1}) \leqslant \frac{2m}{n}[(n_1 - 1)[(n_1 - 1)^2 + 1]]$

Proof. Let $G = K_{1,n_1-1}$ be a fuzzy star graph with $V(G) = (v_0 : v_1, v_2, v_3 \dots v_{n_1-1})$ .Then $d(v_0)=\sum_{i=1}^{n_1-1} \mu_i$ and $d(v_i)=\mu_i$ for $i = 1,2,3 \dots (n_1 - 1)$.

$$\therefore F^*(K_{1,n_1-1}) = \frac{2m}{n}\sum_{i=1}^{n_1} [\sigma(v_i)d(v_i)]^3$$
$$= \frac{2m}{n}[[\sigma(v_0)d(v_0)]^3 + \sum_{i=1}^{n_1-1} [\sigma(v_i)d(v_i)]^3]$$
$$= \frac{2m}{n}[[\sigma(v_0)\sum_{i=1}^{n_1-1} \mu_i]^3 + \sum_{i=1}^{n_1-1} [\sigma(v_i)\mu_i]^3]$$
$$\leq \frac{2m}{n}[(n_1 - 1)^3 + (n_1 - 1)]$$
$$= \frac{2m}{n}[(n_1 - 1)[(n_1 - 1)^2 + 1]]$$

Theorem 4 *Let* G *be* n_1*-vertex fuzzy graph with* m_1*-edges. Then*

$$F^*(G^{++}) \leq \frac{16m\prime}{n\prime}[FI(G) + m_1]$$

Proof. Let $G = (V, \sigma, \mu)$ be any fuzzy graph. Then we have

$$F^*(G^{++}) = \frac{2m\prime}{n\prime}\sum_{u \in V(G^{++})} [\sigma(u)d_G^{++}(u)]^3$$
$$= \frac{2m\prime}{n\prime}[\sum_{u \in V(G^{++}) \cap V(G)} [\sigma(u)d_G(u)]^3 + \sum_{u \in V(G^{++})-V(G)} [\sigma(u)d_G(u)]^3]$$

By Lemma 1, we have

$$F^*(G^{++}) = \frac{2m\prime}{n\prime}[\sum_{u \in V(G)} [\sigma(u)d_G(u)]^3 + \sum_{u \in V(G^{++})-V(G)} [\sigma(u)d_G(u)]^3]$$
$$= \frac{2m\prime}{n\prime}[\sum_{u \in V(G)} [\sigma(u)[2d_G(u)]]^3 + \sum_{e \in E(G)} [\sigma(u)2\mu(e)]]^3$$
$$F^*(G^{++}) = \frac{16m\prime}{n\prime}[FI(G) + \sum_{e \in E(G)} [\sigma(u)\mu(e)]^3]$$

since $0 \leq \sigma(u) \leq 1, 0 \leq \mu(e) \leq 1$

$$F^*(G^{++}) \leq \frac{16m\prime}{n\prime}[FI(G) + m_1]$$

Theorem 5 *Let* G *be* n_1*-vertex fuzzy graph with* m_1*-edges. Then*

$$F^*(G^{+-}) \leq \frac{2m\prime}{n\prime}[FI(G) + 3(m_1 - k)M_1(G) + (m_1 - k)^2[n_1(m_1 - k) + 6m_1] + m_1(m_1 - k)^3],$$

for every $u \in V(G), u \nsim e = m_1 - k$, where $k = deg(u)$.

Proof. Let $G = (V, \sigma, \mu)$ be any fuzzy graph. Then we have

$$F^*(G^{+-}) = \frac{2m\prime}{n\prime}\sum_{u \in V(G^{+-})} [\sigma(u)d_G^{+-}(u)]^3$$
$$= \frac{2m\prime}{n\prime}[\sum_{u \in V(G^{+-}) \cap V(G)} [\sigma(u)d_G(u)]^3 + \sum_{u \in V(G^{+-})-V(G)} [\sigma(u)d_G(u)]^3]$$

By Lemma 1, we have

$$F^*(G^{+-}) = \frac{2m\prime}{n\prime}[\sum_{u \in V(G)} [\sigma(u)d_G(u)]^3 + \sum_{u \in V(G^{+-})-V(G)} [\sigma(u)d_G(u)]^3]$$
$$= \frac{2m\prime}{n\prime}[\sum_{u \in V(G)} [\sigma(u)[d_G(u) + \sum_{u \nsim e} \wedge [\sigma(u), \mu(e)]]^3$$
$$+ \sum_{v \in V(G^{+-})-V(G)} [\sigma(v)\sum_{v \nsim e} [\wedge [\mu(e), \sigma(v)]]^3]$$
$$= \frac{2m\prime}{n\prime}[I_1 + I_2]$$

consider,

$$I_1 = \sum_{u \in V(G)} [\sigma(u)[d_G(u) + \sum_{u \nsim e} \wedge [\sigma(u), \mu(e)]]]^3$$
$$= \sum_{u \in V(G)} [\sigma(u)[d_G(u) + (m_1 - k) \wedge [\sigma(u), \mu(e)]]]^3$$

since $0 \leq \sigma(u) \leq 1, 0 \leq \mu(e) \leq 1$

$$= \sum_{u \in V(G)} [\sigma(u)[d_G(u) + (m_1 - k)]]^3$$

$$= \sum_{u \in V(G)} [[\sigma(u)d(u)]^3 + [\sigma(u)(m_1 - k)]^3 + 3\sigma^3(u)[d(u)(m_1 - k)][d(u) + (m_1 - k)]]$$
$$= FI(G) + n_1(m_1 - k)^3 + X$$

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Also,

$$X = \sum_{u\in V(G)} 3\sigma^3(u)[d(u)(m_1-k)][d(u)+(m_1-k)]]$$
$$= 3(m_1-k)\sigma(u)\sum_{u\in V(G)}[\sigma^2(u)d^2(u)] + 3(m_1-k)^2\sum_{u\in V(G)}[\sigma^3(u)d(u)]$$
$$= 3(m_1-k)M_1(G) + 3(m_1-k)^2(2m_1)$$

Thus

$$I_1 \le FI(G) + 3(m_1-k)M_1(G) + (m_1-k)^2[n_1(m_1-k)+6m_1]$$

Now consider

$$I_2 = \sum_{v\in V(G^{+-})-V(G)}[\sigma(v)[\sum_{v*e}[\wedge[\mu(e),\sigma(v)]]^3$$
$$= \sum_{v\in V(G^{+-})-V(G)}[\sigma(v)[(m_1-k)[\wedge[\mu(e),\sigma(v)]]^3$$

since $0 \le \sigma(u) \le 1, 0 \le \mu(e) \le 1$

$$\le \sum_{v\in V(G^{+-})-V(G)}(m_1-k)^3$$
$$= m_1(m_1-k)^3$$
$$\therefore F^*(G^{+-}) = \frac{2m'}{n'}[I_1+I_2]$$

$$\le \frac{2m'}{n'}[FI(G) + 3(m_1-k)M_1(G) + (m_1-k)^2[n_1(m_1-k)+6m_1] + m_1(m_1-k)^3].$$

Theorem 6 *Let G be n_1-vertex fuzzy graph with m_1-edges. Then*

$$F^*(G^{-+}) \le \frac{2m'}{n'}[n_1(n_1-k-1)^3 + FI(G) + 6m_1(n_1-k-1)^2 + 3(n_1-k-1)M_1(G) + 8m_1]$$

Proof. Let $G = (V,\sigma,\mu)$ be any fuzzy graph. Then we have

$$F^*(G^{-+}) = \frac{2m'}{n'}\sum_{u\in V(G^{-+})}[\sigma(u)d_G^{-+}(u)]^3$$
$$= \frac{2m'}{n'}[\sum_{u\in V(G^{-+})\cap V(G)}[\sigma(u)d_G(u)]^3 + \sum_{u\in V(G^{-+})-V(G)}[\sigma(u)d_G(u)]^3]$$

$$= \frac{2m'}{n'}[\sum_{u\in V(G)}[\sigma(u)[\sum_{v_i*v_j}\wedge[\sigma(v_i),\sigma(v_j)] + d_G(v_i)]]^3 + \sum_{u\in V(G^{-+})-V(G)}[\sigma(u)2\mu(e)]^3]$$
$$= \frac{2m'}{n'}[\sum_{u\in V(G)}[\sigma(u)[(n_1-k-1)+d(u)]]^3 + 8m_1]$$

Now consider

$$I_1 = \sum_{u\in V(G)}[\sigma(u)[(n_1-k-1)+d(u)]]^3$$

$$= \sum_{u\in V(G)}[[\sigma(u)(n_1-k-1)]^3 + [\sigma(u)d(u)]^3 + 3\sigma^3(u)(n_1-k-1)d(u)[(n_1-k-1)+d(u)]]$$
$$\le n_1(n_1-k-1)^3 + FI(G) + X$$

Also,

$$X = \sum_{u\in V(G)} 3\sigma^3(u)[d(u)(n_1-k-1)][d(u)+(n_1-k-1)]]$$

$$= 3(n_1-k-1)^2\sum_{u\in V(G)}[\sigma^3(u)d(u)] + 3(n_1-k-1)\sigma(u)\sum_{u\in V(G)}[\sigma^2(u)d^2(u)]$$
$$\le 3(n_1-k-1)^2(2m_1) + 3(n_1-k-1)M_1(G)$$
$$\le 6m_1(n_1-k-1)^2 + 3(n_1-k-1)M_1(G)$$
$$\therefore I_1 \le n_1(n_1-k-1)^3 + FI(G) + 6m_1(n_1-k-1)^2 + 3(n_1-k-1)M_1(G)$$

Thus

$$F^*(G^{-+}) \le \frac{2m'}{n'}[n_1(n_1-k-1)^3 + FI(G) + 6m_1(n_1-k-1)^2 + 3(n_1-k-1)M_1(G) + 8m_1].$$

Theorem 7 *Let G be n_1-vertex fuzzy graph with m_1-edges. Then*

$$F^*(G^{--}) \le \frac{2m'}{n'}[n_1(n_1+m_1-2k-1)^3 + m_1(m_1-k)^3]$$

Proof. Let $G = (V,\sigma,\mu)$ be any fuzzy graph. Then we have

$$F^*(G^{--}) = \frac{2m'}{n'}\sum_{u\in V(G^{--})}[\sigma(u)d_G^{--}(u)]^3$$

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$$= \frac{2m'}{n'}\left[\sum_{u\in V(G^{--})\cap V(G)} [\sigma(u)d_G(u)]^3 + \sum_{u\in V(G^{--})-V(G)} [\sigma(u)d_G(u)]^3\right]$$
$$= \frac{2m'}{n'}\left[\sum_{u\in V(G))} [\sigma(u)[\sum_{v_i * v_j} \wedge [\sigma(v_i),\sigma(v_j)] + \sum_{v*e} \wedge [\sigma(v),\mu(e)]]]^3\right.$$
$$\left.+ \sum_{u\in V(G^{--})-V(G)} [\sigma(v) \sum_{v*e} \wedge [\sigma(v),\mu(e)]]^3\right]$$
$$= I_1 + I_2.$$

consider

$$I_1 = \sum_{u\in V(G))} [\sigma(u)[\sum_{v_i * v_j} \wedge [\sigma(v_i),\sigma(v_j)] + \sum_{v*e} \wedge [\sigma(v),\mu(e)]]]^3$$

since $0 \le \sigma(u) \le 1, 0 \le \mu(e) \le 1$

$$\le \sum_{u\in V(G))} [(n_1 - k - 1)(1) + (m_1 - k)(1)]^3$$
$$= n_1[(n_1 - k - 1) + (m_1 - k)]^3$$
$$= n_1[(n_1 + m_1 - 2k - 1)]^3$$

similarly $I_2 \le m_1(m_1 - k)^3$

Thus

$$F^*(G^{--}) \le \frac{2m'}{n'}[n_1(n_1 + m_1 - 2k - 1)^3 + m_1(m_1 - k)^3].$$

Conclusion:

In this paper we have studied the F^* index of fuzzy generalized transformation graphs and obtained some upper bounds for $F^*(G)$ in terms of elements of a graph G.

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