# A COMPARATIVE STUDY ON ADVANCEMENTS IN NEUTROSOPHIC NUMBER THEORETIC CONCEPTS IN THE NEUTROSOPHIC RING OF INTEGERS AND REFINED NEUTROSOPHIC RING OF INTEGERS

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# A COMPARATIVE STUDY ON ADVANCEMENTS IN NEUTROSOPHIC NUMBER THEORETIC CONCEPTS IN THE NEUTROSOPHIC RING OF INTEGERS AND REFINED NEUTROSOPHIC RING OF INTEGERS

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**Abstract:** *The study of neutrosophic numbers has garnered significant attention due to their ability to address uncertainty, indeterminacy, and contradictions in various mathematical models. Number-theoretic concepts within the Neutrosophic Ring of Integers $Z(I)$ are explored, including conditions for division, Euler's function, and congruences, along with other classical concepts. Additionally, further number-theoretic principles, such as division, congruences, and the existence of solutions to linear Diophantine equations, are examined and reviewed in the context of the Refined Neutrosophic Ring of Integers. This paper presents a comparative study between the neutrosophic ring of integers (NRI) and the refined neutrosophic ring of integers (RNRI).*

**Keywords:** *Neutrosophic ring of integers, refined neutrosophic ring of integers, neutrosophic numbers, uncertainty modeling, number theory, indeterminacy, mathematical structures.*

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## 1. Introduction

Mathematical structures such as rings and fields have long played a pivotal role in number theory, offering foundational frameworks for solving equations and modeling various phenomena. However, traditional structures often assume complete certainty and precision, which can be limiting when dealing with real-world problems that involve uncertainty, ambiguity, or incomplete information. Neutrosophic numbers, introduced in [14], offer a flexible way to extend classical mathematical concepts by incorporating three parameters: truth-membership (T), indeterminacy-membership (I), and falsity-membership (F).

This framework has been applied across various fields, including mathematics and physics, to address complex problems involving uncertainty. The integration of algebraic structures into neutrosophy was pioneered by Kandasamy and Smarandache in [8],[9], marking a significant development in the application of neutrosophic theory.

Neutrosophy, has found significant applications across various fields such as computing, decision-making, medical research, and applied sciences. This framework offering a more nuanced approach to dealing with imprecise and inconsistent data. Neutrosophic theory significantly enhances decision-making processes by providing a robust framework for handling ambiguity and uncertainty. It introduces aggregation operators and information measures that improve decision-making in complex scenarios [6].

The bibliometric study highlights the growing interest and application of neutrosophic theory in decision-making, underscoring its importance in contemporary research. In medical image processing, neutrosophic logic optimizes tasks such as noise reduction, image segmentation, and feature extraction. For example, the peak signal-to-noise ratio of MRI images increased from 25 dB to 35 dB, and the accuracy of breast cancer detection improved from 88% to 95% using neutrosophic logic. These improvements demonstrate the potential of neutrosophic logic to enhance the accuracy and reliability of medical diagnostics [13].

Neutrosophic statistics offer a new paradigm for handling indeterminate and vague information, providing a more realistic representation of data. This approach is particularly useful in situations where traditional statistical methods fall short, thereby improving the reliability of analytical results [1].

The flexibility of neutrosophic sets in handling vagueness and contradiction makes them applicable in various real-world scenarios, including pattern recognition and expert systems [15].

Neutrosophic algebraic structures provide a foundational framework for extending classical mathematical concepts into the neutrosophic realm. These structures are essential for developing models that incorporate indeterminacy, as demonstrated in the application of neutrosophic graphs and cognitive maps [16].

The concept of neutrosophic triplet groups (NTGs) further generalizes standard group theory by introducing elements with unique neutral elements, allowing for the exploration of generalized symmetry and subgroup properties [17].

Neutrosophic number theory is an innovative extension of classical number theory, integrating the principles of neutrosophic logic to accommodate indeterminacy alongside truth and falsehood. This approach allows for a more nuanced exploration of numerical properties, such as integers, divisibility, and prime numbers, by incorporating uncertainty and ambiguity. The development of neutrosophic number theory is supported by various mathematical structures and models that enhance its applicability in complex decision-making and problem-solving scenarios.

The exploration of divisibility within the framework of neutrosophic integers extends classical number theory into a domain where indeterminacy is a central element. This approach has significant implications for logic, decision-making, and artificial intelligence. The concept of the greatest common divisor (GCD) in neutrosophic integers is pivotal, as it aligns with traditional number theory while accommodating indeterminacy. This extension also includes classical theorems like Euler's theorem, which remains valid in this new setting, and the study of equations such as Pell's equation, which now incorporate indeterminate coefficients and solutions. [[3],[7],[10], [12]]

The introduction of neutrosophic numbers allows for the representation of indeterminate information, which is crucial for decision-making processes that require handling uncertainty. [11]

The potential applications of neutrosophic number theory in modern cybernetic systems are highlighted in [5]. Also, how neutrosophic theory can be applied to improve current security algorithms are examined. In cryptography, both frameworks have potential applications in developing encryption algorithms that can handle uncertainty and ambiguous data more effectively.

The neutrosophic ring of integers (NRI) is an extension of the classical ring of integers, where the operations of addition and multiplication are performed in the neutrosophic set framework. In recent years, further advancements have led to the development of the refined neutrosophic ring of integers (RNRI), where the indeterminacy parameter is more rigorously defined, offering a more nuanced approach to handling uncertainty. This review paper aims to provide a comparative analysis of the NRI and RNRI, highlighting key differences in their theoretical foundations, structural properties based on the work presented in [4], [10],[12].

## 2. Main Discussion
### 2.1. Preliminaries

The concept of a neutrosophic ring of integers is rooted in neutrosophic set theory, where elements are characterized by three components: T (truth), I (indeterminacy), and F (falsity). These values lie within the interval [0, 1], and operations such as addition and multiplication are extended to accommodate this tripartite structure. In the NRI, each integer is associated with a neutrosophic number, allowing for the representation of uncertainty within arithmetic operations.

The primary advantage of the NRI is its ability to model uncertainty in algebraic operations, which is particularly useful in fields such as decision theory and cryptography, where the presence of indeterminate information can affect outcomes.

### Definition 2.1.1: [3]

Given any ring $(R, +, .)$ and introducing an indeterminacy element I with the property $I^2 = I$, the set $R[I] = \{a + bI: a, b \in R\}$ is called a neutrosophic ring generated by $I$ and $R$ under the binary operations of $R$.

If $R = Z$ is the ring of integers, then $Z(I) = \{a + bI: a, b \in Z\}$ is called the neutrosophic ring of integers. Elements of Z(I) are called neutrosophic integers.

### Definition 2.1.2: [2]

Let $R(I) = \{a + bI; a, b \in R\}$ be the real neutrosophic field, where R is a field of reals. For $a + bI, c + dI \in R[I]$, we say that $a + bI \le c + dI$ if and only if $a \le c$ and $a + b \le c + d$.

### Definition 2.1.3:

Let $R[I] = \{a + bI: a, b \in R\}$ be the field of neutrosophic real numbers where $R$ is the field of real numbers. $a + bI \in R[I]$ is said to be positive neutrosophic real numbers $iff\ a \ge 0, a + b \ge 0$.

For example, $x = 3 - 2I$ is a neutrosophic positive real number, since $3 \ge 0$ and $(3 - 2) = 1 \ge 0$. Similar way, $x = 2 + I$ is a neutrosophic positive real number.

### Definition 2.1.4:

Let $R(I) = \{a + bI; a, b \in R\}$ be the real neutrosophic field. For $a + bI, c + dI \in R(I)$ the arithmetic operation addition is defined as, $(a + bI) + (c + dI) = (a + c) + (b + d)I$

### Definition 2.1.5:

Let $R(I) = \{a + bI; a, b \in R\}$ be the real neutrosophic field. For $a + bI, c + dI \in R(I)$ the arithmetic operation multiplication is defined as, $(a + bI)(c + dI) = ac + (ad + bc + bd)I$

## 2.2. Number Theory in the Neutrosophic Ring of Integers (NRI) $Z(I)$

The preliminary concepts and theorems about Neutrosophic rings of integers that were introduced in the research papers [2], [3], [4], and [7] are reviewed and summarized in this section.

### Definition 2.2.1: (Division)

Let $Z(I) = \{a + bI; a, b \in Z\}$ be the neutrosophic ring of integers. For any $x, y \in Z(I)$, we say that $x \mid y$ if $\exists\ r \in Z(I)$ such that $r \cdot x = y$.

### Theorem 2.2.1: (Form of division in $Z(I)$ )

Let $Z(\mathrm{I}) = \{a + bI; a, b \in Z\}$ the neutrosophic ring of integers, $x = x_1 + x_2I, y = y_1 + y_2I$ be two arbitrary elements in Z(I). Then $x \mid y$ if and only if $x_1 \mid y_1$ and $x_1 + x_2 \mid y_1 + y_2$.

Proof: Suppose that $x \mid y$, hence there is $r = r_1 + r_2 I \in Z(I); r.x = y$. This implies
(I) $r_1 x_1 = y_1$, i.e. $x_1 \mid y_1$
(II) $r_1 x_2 + r_2 x_1 + r_2 x_2 = y_2$.

By adding (I) to (II) we get
$r_1 x_1 + r_1 x_2 + r_2 x_1 + r_2 x_2 = y_1 + y_2$,

which implies that

$(r_1 + r_2)(x_1 + x_2) = y_1 + y_2$.

Thus $x_1 + x_2 \mid y_1 + y_2$.

Conversely, assume that $x_1 \mid y_1$ and $x_1 + x_2 \mid y_1 + y_2$, hence there is $a, b \in Z$ such that $a x_1 = y_1$ and $b(x_1 + x_2) = y_1 + y_2$. We put $r = a + (b - a)I$.

then we get $r \cdot x = y$ and $x \mid y$.

**Definition 2.2.2: (primes)**

Let $Z(I) = \{a + bI; a, b \in Z\}$ the neutrosophic ring of integers. An arbitrary element $x \in Z(I)$ is called prime if $x \mid y.z$ implies $x \mid y$ or $x \mid z$, where $y, z \in Z(I)$.

**Theorem 2.2.2: (Form of primes in $Z(I)$) [7]**

Let $Z(I) = \{a + bI; a, b \in Z\}$ the neutrosophic ring of integers. Then primes in $Z(I)$ have one of the following forms: $x = \pm p + (\pm 1 \pm p) I$ or $x = \pm 1 + (\pm p \pm 1)I; p$ is any prime in $Z$.

**Definition 2.2.3: (Congruence)**

Let $x = a + bI, y = c + dI, z = m + nI$ be three elements in $Z(I)$. We say that $x \equiv y (\mathrm{mod} z)$ if and only if $z \mid x - y$.

**Definition 2.2.4: (Greatest Common Divisor)**

An element $z$ in $Z(I)$ is called the greatest common divisor (GCD) of $x$ and $y$ if it satisfies the following conditions:

1. Divisibility: $z$ divides both $x$ and $y$: $z \mid x$ and $z \mid y$.

2. Maximality: For any other element $c$ dividing both $x$ and $y$ (i.e., $c \mid x$ and $c \mid y$), $c$ must also divide $z$ (i.e., $c \mid z$). (b) We say that $z = \gcd(x, y)$ if and only if $z \mid x$ and $z \mid y$ and for each divisor $c \mid x$ and $c \mid y$, then $c \mid z$. $x, y$ are called relatively prime in $Z(I)$ if and only if $\gcd(x, y) = 1$.

**Theorem 2.2.3: (Form of congruencies in $Z(I)$) [4]**

Let $x = a + bI, y = c + dI, z = m + nI$ be three elements in $Z(I)$. Then $x \equiv y (\mathrm{mod} z)$ if and only if $a \equiv c (\mathrm{mod} m), a + b \equiv c + d (\mathrm{mod}\ m + n)$.

**Proof:** We suppose that $x \equiv y (\mathrm{mod} z)$, hence $z \mid x - y$, i.e. $m + nI \mid (a - c) + (b - d)I$. This implies $m \mid a - c$ and $m + n \mid (a + b) - (c + d)$, thus $a \equiv c (\mathrm{mod} m), a + b \equiv c + d (\mathrm{mod} m + n)$.

Conversely, we suppose that $a \equiv c (\mathrm{mod} m), a + b \equiv c + d (\mathrm{mod} m + n)$, hence $m \mid a - c$ and $m + n \mid (a + b) - (c + d)$, this implies that $m + nI \mid (a - c) + (b - d)I$, i.e. $z \mid x - y$, which means that $x \equiv y (\mathrm{mod} z)$.

**Example 2.2.1:** $3 + 5I \equiv (1 + 3I)(mod2 + 2I)$. This is because $3 \equiv 1(mod2)$,

$3 + 5 = 8 \equiv 1 + 3 = 4(mod4)$.

$8 + 7I \equiv (2 + 3I)(mod3 + 2I)$. This is because

$8 \equiv 2(mod3), 8 + 7 = 15 \equiv 2 + 3 = 5(mod5)$.

**Theorem 2.2.4: [4]** Let $x = a + bI, y = c + dI, z = m + nI$ be three elements in $Z(I)$. Then $z = \gcd(x, y)$ if $m = \gcd(a, c)$ and $m + n = \gcd(a + b, c + d)$.

**Proof:**
Consider $z = m + nI$, where $m = \gcd(a, c)$ and $m + n = \gcd(a + b, c + d)$. It is easy to check that $z \mid x$ and $z \mid y$, that is because $m = \gcd(a, c)|a, m = \gcd(a, c)|c$, and $m + n = \gcd(a + b, c + d)|a + b, m + n = \gcd(a + b, c + d)|c + d$. On the other hand, we assume that $l = f + gI$ is a common divisor of $x$ and $y$. We shall prove that $l \mid z$.

Since $l$ is a common divisor, then we have $f \mid a$ and $f \mid c$, hence $f \mid \gcd(a, c) = m$. Also, we have $f + g \mid a + b$ and $f + g \mid c + d$, hence $f + g \mid \gcd(a + b, c + d) = m + n$. This implies that $l \mid z$, and $z = \gcd(x, y)$.

**Example 2.2.2:** $gcd(8 + 7I, 2 + 3I) = 1 + 4I$, that is because $gcd(8,1) = 1 = m$,

$gcd(8 + 7, 2 + 3) = gcd(15, 5) = 5 = m + n$,

thus $m + nI = 1 + 4I = gcd(3 + 5I, 1 + 3I)$.

**Theorem 2.2.5: (Euclidian division theorem in Z(I))**

Let Z(I) be the neutrosophic ring of integers, $x = a + bI, y = c + dI$ be two arbitrary elements in Z(I). There are two elements $q = s + tI, r = m + nI$ such that $x = q \cdot y + r$.

Proof:
By the division theorem in Z, we can find the following integers: $q_1, q_2, r_1, r_2: a = q_1 c + r_1$, and $a + b = (c + d)q_2 + r_2$. By putting $s = q_1, t = (q_2 - q_1), m = r_1, n = (r_2 - r_1)$, we find that $x = q \cdot y + r$.

**Remark 2.2.1:** The uniqueness of $q$ and $r$ is well-defined in classical number theory but in case Neutrosophic Ring of Integers that uniqueness is not defined yet.

**Example 2.2.3:** Consider the following neutrosophic integers $x = 7 + 5I, y = 2 + 2I$. There are $q = 1 + I, r = 5 - I$ such that $x = q \cdot y + r$.

**Remark 2.2.2:** (Solvability of a linear congruence in Z(I) )

To solve a linear congruence $x + yI \equiv a + bI(\mod m + nI)$, we can transform it into equivalent congruences using Theorem 2.3:

$x \equiv a(\mod m)$, and $x + y \equiv (a + b)(\mod m + n)$. Then, we solve the resulting system of congruences and compute $x, y$.

**Example 2.2.4:** Consider the following neutrosophic linear congruence $x + yI \equiv 2 + 5I(\mod 3 + I)$. Its equivalent system can be expressed as:

(a) $x \equiv 2 \ (\mathrm{mod}\, 4)$. (It has a solution $x = 2$).
(b) $x + y \equiv 7 \ (\mathrm{mod}\ 4)$. (It has a solution $x + y = 3$, hence $y = 1$. Therefore, $2 + I$ is a solution of the original neutrosophic congruence.

**Definition 2.2.5:** Neutrosophic Exponentiation

Let $a + bI, c + dI$ be two elements in $Z(\mathrm{I})$ then

$$(a + bI)^{c+dI} = a^c + I[(a + b)^{c+d} - a^c]$$

**Definition 2.2.6:** (Euler's function in $Z(\mathrm{I})$ )

The neutrosophic Euler's function on $Z(I)$ is defined as:

$$\varphi(a + bI) = |\{x = c + dI; \gcd(c + dI, a + bI) = 1\}|, \text{ where } c + dI \leq a + bI.$$

i.e. $\varphi(a + bI)$ counts the number of positive neutrosophic integers not exceeding $a + bI$ that are relatively prime to $a + bI$.

**Theorem 2.2.6:** (Euler's Theorem in $Z(I)$ )

(a) Let $x = a + bI$ be any element in $Z(I)$, then $\varphi(x) = \varphi(a) \times \varphi(b + a), a, a + b > 0$.
(b) If $y = c + dI$ is a neutrosophic integer with $\gcd(x, y) = 1$, hence $y^{\varphi(x)} \equiv 1 (\mathrm{mod}\, x)$. (neutrosophic Euler's Theorem).

Proof: (a) Let $y = c + dI$ be any neutrosophic integer with, $c + dI \leq a + bI$, and $\gcd(x, y) = 1$. We can see by Theorem 2.4 that $\gcd(a, c) = 1, \gcd(a + b, c + d) = 1$, i.e. $a, c$ are relatively prime and $a + b, c + d$ are relatively prime, hence we get that $\varphi(x) = \varphi(a) \times \varphi(b + a)$.

(b) According to Euler's Theorem, we have

$c^{\varphi(a)} \equiv 1 (\mathrm{mod}\ a)$, and $(c + d)^{\varphi(a+b)} \equiv 1 (\mathrm{mod}\ a + b)$, since $\gcd(a, c) = \gcd(a + b, c + d) = 1$ under the assumption of $\gcd(x, y) = 1$.

Now, we can write $c^{\varphi(a) \times \varphi(b+a)} = c^{\varphi(x)} \equiv 1 (\mathrm{mod}\ a)$,

$(c + d)^{\varphi(a) \times \varphi(b+a)} = (c + d)^{\varphi(x)} \equiv 1 (\mathrm{mod}\ a + b)$.

Now, we compute $y^{\varphi(x)} = (c + dI)^{\varphi(x)}$

$$= c^{\varphi(x)} + I \left[ \sum_{i=1}^{\varphi(x)} \binom{\varphi(x)}{i} c^{\varphi(x)-i} d^i \right] \text{ (using Definition 2.5)}$$

$$= c^{\varphi(x)} + I[(c + d)^{\varphi(x)} - c^{\varphi(x)}] = m + nI.$$

We remark that $m = c^{\varphi(x)} \equiv 1 (\mathrm{mod}\ a), m + n = (c + d)^{\varphi(x)} \equiv 1 (\mathrm{mod}\ a + b)$, this implies that $y^{\varphi(x)} = m + nI \equiv 1 \ (\mathrm{mod}\ a + bI)$.

**Example 2.2.5:** For $x = 3 + 5I, \varphi(x) = 8 \ and$ these neutrosophic integers are

$1 + 0I, 1 + 2I, 1 + 4I, 1 + 6I, 2 - I, 2 + I, 2 + 3I, 2 + 5I.$

Also, $\varphi(3 + 5I) = \varphi(3) \times \varphi(8) = 2 \times 4 = 8.$

**Definition 2.2.7:** Linear Diophantine equation in $Z(I)$

Let $Z(I) = \{a + bI; a, b \in Z\}$ be the neutrosophic ring of integers. The neutrosophic linear Diophantine equation with two variables is defined as follows:

$AX + BY = C; \; A\,B, C \in Z(I)$.

**Theorem 2.2.7:** Let $Z(I) = \{a + bI; a, b \in Z\}$ be the neutrosophic ring of integers. The neutrosophic linear Diophantine equation $AX + BY = C$ with two variables $X = x_1 + x_2I$, $Y = y_1 + y_2I$, where $A = a_1 + a_2I$, $B = b_1 + b_2I$ is equivalent to the following two classical Diophantine equations:

(1) $a_1x_1 + b_1y_1 = c_1$.
(2) $(a_1 + a_2)(x_1 + x_2) + (b_1 + b_2)(y_1 + y_2) = c_1 + c_2$

Proof: It is sufficient to show that $AX + BY = C$ implies (1) and (2).
$AX + BY = C$ is equivalent to:

$(a_1 + a_2I)(x_1 + x_2I) + (b_1 + b_2I)(y_1 + y_2I) = c_1 + c_2I$, by easy computing we find
$[a_1x_1 + b_1y_1] + [a_1x_2 + a_2x_1 + a_2x_2 + b_1y_2 + b_2y_1 + b_2y_2]I = c_1 + c_2I$, hence
$a_1x_1 + b_1y_1 = c_1$, and $a_1x_2 + a_2x_1 + a_2x_2 + b_1y_2 + b_2y_1 + b_2y_2 = c_2$. We can see that we get equation (1). For equation (2) we take

$a_1x_2 + a_2x_1 + a_2x_2 + b_1y_2 + b_2y_1 + b_2y_2 = c_2$,

By adding equation (1) to both sides of this equation, we have

$a_1x_1 + b_1y_1 + a_1x_2 + a_2x_1 + a_2x_2 + b_1y_2 + b_2y_1 + b_2y_2 = c_1 + c_2$,

which implies that $(a_1 + a_2)(x_1 + x_2) + (b_1 + b_2)(g_1 + y_2) = c_1 + c_2$

i.e. the equation (2) is obtained.

**Theorem 2.2.8:** (The criteria for the solvability of neutrosophic Linear Diophantine equation)

Let $Z(I) = \{a + bI; a, b \in Z\}$ be the neutrosophic ring of integers. The neutrosophic linear Diophantine equation $AX + BY = C$ with two variables $X = x_1 + x_2I, Y = y_1 + y_2I$ and $A = a_1 + a_2I, B = b_1 + b_2I$ is solvable if and only if $gcd(a_1, b_1)|c_1, gcd(a_1 + a_2, b_1 + b_2)|c_1 + c_2$.

**Example 2.2.6:** Consider the neutrosophic linear Diophantine equation

$(2 + 2I)X + (3 + 4I)Y = 5 + 5I$ \hfill (1)

where $X = x_1 + x_2I, Y = y_1 + y_2I$

Equation (1) is equivalent to:

$2x_1 + 3y_1 = 5$ and $(2 + 2)(x_1 + x_2) + (3 + 4)(y_1 + y_2) = 5 + 5$.

Now, $2x_1 + 3y_1 = 5$ is a classical linear Diophantine equation. It has a solution $x_1 = 4, y_1 = -1$.

$(2 + 2)(x_1 + x_2) + (3 + 4)(y_1 + y_2) = 5 + 5$, i.e $4M + 7N = 10; M = x_1 + x_2, N = y_1 + y_2$. It is a classical linear Diophantine equation with $M, N$ as variables. It has a solution $M = -1, N = 2$. $x_2 = M - x_1 = -5, y_2 = N - y_1 = 3$, thus, the equation (1) has a solution $X = 4 - 5I, Y = -1 + 3I$.

### 2.3. The Refined Neutrosophic Ring of Integers (RNRI)

The refined neutrosophic ring of integers builds upon the NRI by providing a more detailed characterization of the indeterminacy parameter. In the RNRI, indeterminacy is not treated as a single value but is broken down into multiple subcategories, such as partial indeterminacy and total indeterminacy. This refined approach allows for greater precision in scenarios where the degree of uncertainty varies across different operations.

This section provides a summary of the fundamental concepts and findings presented in the research work [10].

**Definition 2.3.1:** Let $(R, +, \times)$ be a ring, $(R(I_1, I_2), +, X)$ is called a refined neutrosophic ring generated by R, $I_1, I_2$.

If $R = Z$, then $(R(I_1, I_2), +, X)$ is called the refined neutrosophic ring of integers.

**Remark 2.3.1:** The element I can be split into two indeterminacies $I_1, I_2$ with conditions:

$$I_1{}^2 = I_1, I_2{}^2 = I_2, I_1.I_2 = I_2.I_1 = I_1.$$

**Definition 2.3.2: (Division)**

Let $Z(I_1, I_2) = \{(a, bI_1, cI_2); a, b, c \in Z\}$ be the refined neutrosophic ring of integers. For any $x, y \in Z(I_1, I_2)$ we say that $x|y$ if there is $r \in Z(I_1, I_2); r. x = y$.

**Theorem 2.3.1: (Form of division in $Z(I_1, I_2)$)**

Let $Z(I_1, I_2) = \{( a, bI_1, cI_2); a, b, c \in Z\}$ the refined neutrosophic ring of integers, $x = (x_0, x_1 I_1, x_2 I_2)$, $y = (y_0, y_1 I_1, y_2 I_2)$ be two arbitrary elements in $Z(I_1, I_2)$. Then $x|y$ if and only if $x_0|y_0$, $x_0 + x_2|y_0 + y_2$, $x_0 + x_1 + x_2|y_0 + y_1 + y_2$.

**Proof:** Suppose that $x|y$ in $Z(I_1, I_2)$, then there is $r = (r_0, r_1 I_1, r_2 I_2) \in Z(I_1, I_2)$. such that

(1) $r. x = y$

By easy computing to equation (1) we get the following equivalent equations:

(a) $r_0 x_0 = y_0$, i.e. $x_0|y_0$.

(b) $r_0 x_2 + r_2 x_2 + r_2 x_0 = y_2$ .

(c) $r_0 x_1 + r_2 x_1 + r_1 x_0 + r_1 x_1 + r_1 x_2 = y_1$

By adding equation (a) to (b) we get (2) $(r_0 + r_2)( x_0 + x_2) = y_0 + y_2$, i. e. $x_0 + x_2|y_0 + y_2$.

Now, we add equation (2) to (c) to get $(r_0 + r_1 + r_2)( x_0 + x_1 + x_2) = y_0 + y_1 + y_2$ , i. e.

$x_0 + x_1 + x_2|y_0 + y_1 + y_2$.

Conversely, we assume that $x_0|y_0$, $x_0 + x_2|y_0 + y_2$, $x_0 + x_1 + x_2|y_0 + y_1 + y_2$.

There are

$a, b, c \in Z; ax_0 = y_0, b(x_0 + x_2) = y_0 + y_2, c( x_0 + x_1 + x_2) = y_0 + y_1 + y_2$ .

We put

$$r_0 = a, r_2 = b - a, r_1 = c - b.$$

Now, we get $r = (r_0, r_1 I_1, r_2 I_2) \in Z(I_1, I_2)$, $and\ r \cdot x = y$, hence $x|y$.

**Definition 2.3.3:** Let $Z(I_1, l_2) = \{(a, bI_1, cI_2); a, b, c \in Z\}$ be the refined neutrosophic ring of integers. The refined neutrosophic linear Diophantine equation with two variables is defined as follows: $AX + BY = C$, $A, B, C \in Z(I_1, I_2)$.

**Theorem 2.3.2:** Let $Z(I_1, I_2) = \{(a, bI_1, cI_2); a, b, c \in Z\}$ be the refined neutrosophic ring of integers, $AX + BY = C$; $A, B, C \in Z(I_1, I_2)$ be a refined neutrosophic linear Diophantine equation, where $X = (x_0, x_1 I_1, x_2 I_2), Y = (y_0, y_1 I_1, y_2 I_2), A = (a_0, a_1 I_1, a_2 I_2)$, $B = (b_0, b_1 I_1, b_2 I_2)$, $C = (c_0, c_1 I_1, c_2 I_2)$. Then $AX + BY = C$ is equivalent to the following three Diophantine equations:

(1) $a_0 x_0 + b_0 y_0 = c_0$.
(2) $(a_0 + a_2)(x_0 + x_2) + (b_0 + b_2)(y_0 + y_2) = c_0 + c_2$.
(3) $(a_0 + a_1 + a_2)(x_0 + x_1 + x_2) + (b_0 + b_1 + b_2)(y_0 + y_1 + y_2) = c_0 + c_1 + c_2$.

Proof: By replacing $A, B, C, X, Y$ we find
$AX = (a_0, a_1 I_1, a_2 I_2)(x_0, x_1 I_1, x_2 I_2) =$
$(a_0 x_0, [a_0 x_1 + a_1 x_0 + a_1 x_1 + a_1 x_2 + a_2 x_1]I_1, [a_0 x_2 + a_2 x_0 + a_2 x_2]I_2)$,
$BY = (b_0, b_1 I_1, b_2 I_2)(y_0, y_1 I_1, y_2 I_2) =$
$(b_0 y_0, [b_0 y_1 + b_1 y_0 + b_1 y_1 + b_1 y_2 + b_2 y_1]I_1, [b_0 y_2 + b_2 y_0 + b_2 y_2]I_2)$, thus the equation $AX + BY = C$ implies
(*) $a_0 x_0 + b_0 y_0 = c_0$, which is the equation *(1)*.

(**) $a_0 x_2 + a_2 x_0 + a_2 x_2 + b_0 y_2 + b_2 y_0 + b_2 y_2 = c_2$
(***) $a_0 x_1 + a_1 x_0 + a_1 x_1 + a_1 x_2 + a_2 x_1 + b_0 y_1 + b_1 y_0 + b_1 y_1 + b_1 y_2 + b_2 y_1 = c_1$.
By adding (*) to (**) we get $(a_0 + a_2)(x_0 + x_2) + (b_0 + b_2)(y_0 + y_2) = c_0 + c_2$, which is the required Equation (2).
By adding (2) to (***) we get
$(a_0 + a_1 + a_2)(x_0 + x_1 + x_2) + (b_0 + b_1 + b_2)(y_0 + y_1 + y_2) = c_0 + c_1 + c_2$. which is the required Equation (3).

**Theorem 2.3.3:** Let $Z(I_1, I_2) = \{(a, bI_1, cI_2); a, b, c \in Z\}$ be the refined neutrosophic ring of integers, $AX + BY = C$; $A, B, C \in Z(I_1, I_2)$ be a refined neutrosophic linear Diophantine equation, where
$X = (x_0, x_1 I_1, x_2 I_2), Y = (y_0, y_1 I_1, y_2 I_2), A = (a_0, a_1 I_1, a_2 I_2)$,
$B = (b_0, b_1 I_1, b_2 I_2), C = (c_0, c_1 I_1, c_2 I_2)$. Then $AX + BY = C$ is solvable if and only if:
(a) $\gcd(a_0, b_0)|c_0$.
(b) $\gcd(a_0 + a_2, b_0 + b_2) \mid c_0 + c_2$.
(c) $\gcd(a_0 + a_1 + a_2, b_0 + b_1 + b_2)|c_0 + c_1 + c_2$.

The proof is similar to that of Theorem 2.8.

**Example 2.3.1:** (a) Consider the refined neutrosophic linear Diophantine equation
$(2, 3I_1, 4I_2) \cdot X + (4, 4I_1, 7I_2)Y = (4, 6I_1, I_2)$, we have
$\gcd(2,4) = 2|4, \gcd(2 + 4, 4 + 7) = \gcd(6,11) = 1|(4 + 1 = 5)$,
$\gcd(2 + 3 + 4, 4 + 4 + 7) = \gcd(9,15) = 3$ which does not divide $4 + 6 + 1 = 11$, so the refined neutrosophic linear Diophantine equation is not solvable.
(b) Consider the refined neutrosophic linear Diophantine equation
$(2, 3I_1, 4I_2) \cdot X + (4, 4I_1, 7I_2)Y = (4, 6I_1, 2I_2)$, we have
$\gcd(2,4) = 2|4, \gcd(2 + 4, 4 + 7) = \gcd(6,11) = 1|(4 + 1 = 5)$,
$\gcd(2 + 3 + 4, 4 + 4 + 7) = \gcd(9,15) = 3 \mid (4 + 6 + 2 = 12)$. Therefore, it is solvable.

**Example 2.3.2:** In example 3.1 (b), it was observed that $(2,3I_1,4I_2) \cdot X + (4,4I_1,7I_2)Y = (4,6I_1,2I_2)$ is solvable.

We consider $2x_0 + 4y_0 = 4$. It has a solution $x_0 = 4, y_0 = -1$ We take $(2+4)(x_0 + x_2) + (4+7)(y_0 + y_2) = 4 + 2$, i.e $6M + 11N = 6$; $M = x_0 + x_2$, and $N = y_0 + y_2$, it has a solution $M = 1, N = 0$, thus $x_2 = M - x_0 = -3, y_2 = N - y_0 = 1$. The third equation is $(2 + 3 + 4)(x_0 + x_1 + x_2) + (4 + 4 + 7)(y_0 + y_1 + y_2) = 4 + 6 + 2$, i.e $9S + 15T = 12$; $S = x_0 + x_1 + x_2, T = y_0 + y_1 + y_2$. It has a solution $S = 3, T = -1$, thus $x_1 = S - x_0 - x_2 = 2, y_1 = T - y_0 - y_2 = -1$. The solution of $(2,3I_1,4I_2) \cdot X + (4,4I_1,7I_2)Y = (4,6I_1,2I_2)$ is $X = (4, 2I_1, I_2), Y = (-1, -I_1, I_2)$.

**Definition 2.3.4: (Congruence)**

Let $x = (x_0, x_1 I_1, x_2 I_2)$, $y = (y_0, y_1 I_1, y_2 I_2)$, $z = (z_0, z_1 I_1, z_2 I_2)$ be three elements in $Z(I_1, I_2)$. We say that $x \equiv y(mod z)$ if and only if $z|x - y$.

**Theorem 2.3.4: (Form of congruencies in $Z(I_1, I_2)$)**

Let $x = (x_0, x_1 I_1, x_2 I_2)$, $y = (y_0, y_1 I_1, y_2 I_2)$, $z = (z_0, z_1 I_1, z_2 I_2)$ be three elements in $Z(I_1, I_2)$. Then $x \equiv y(mod z)$ if and only if $x_0 \equiv y_0 \ (mod z_0)$, $x_0 + x_2 \equiv y_0 + y_2 \ (mod \ z_0 + z_2)$, $x_0 + x_1 + x_2 \equiv y_0 + y_1 + y_2 \ (mod \ z_0 + z_1 + z_2)$.

Proof: We assume that $x \equiv y(mod z)$, then $z|x - y$. By Theorem 3.2, we find that $z_0|x_0 - y_0$, $(z_0 + z_2)|(x_0 + x_2) - (y_0 + y_2)$, $(z_0 + z_1 + z_2)|(x_0 + x_1 + x_2) - (y_0 + y_1 + y_2)$, thus

$x_0 \equiv y_0 \ (mod z_0)$, $x_0 + x_2 \equiv y_0 + y_2 \ (mod \ z_0 + z_2)$, $x_0 + x_1 + x_2 \equiv y_0 + y_1 + y_2 \ (mod \ z_0 + z_1 + z_2)$.

Conversely, $x_0 \equiv y_0 \ (mod z_0)$, $x_0 + x_2 \equiv y_0 + y_2 \ (mod \ z_0 + z_2)$, $x_0 + x_1 + x_2 \equiv y_0 + y_1 + y_2 \ (mod \ z_0 + z_1 + z_2)$.

Then for some $a, b, c \in Z, x_0 - y_0 = az_0$, $(x_0 + x_2) - (y_0 + y_2) = b(z_0 + z_2)$,

$(x_0 + x_1 + x_2) - (y_0 + y_1 + y_2) = c(z_0 + z_1 + z_2)$,

Putting $r_0 = a, r_2 = b - a, r_1 = c - b$.

Now, we get $r = (r_0, r_1 I_1, r_2 I_2) \in Z(I_1, I_2), and \ r.z = x - y$, hence $x \equiv y(mod \ z)$.

Hence the proof.

**Example 2.3.3:** $(5, I_1, 2I_2) \equiv (3, I_1, 4I_2)\big(mod(2, -I_1, I_2)\big)$, that is because
$5 \equiv 3(mod \ 2), 5 + 2 = 7 \equiv 7 = (3 + 4)(mod \ 3), 5 + 1 + 2 = 8 \equiv (3 + 1 + 4)(mod \ 3)$.

**Theorem 2.3.5:** Let $x = (x_0, x_1 I_1, x_2 I_2), y = (y_0, y_1 I_1, y_2 I_2)$ be two elements in $Z(I_1, I_2)$. Then
$r = \gcd(x, y) = (m, n I_1, t I_2); m = \gcd(x_0, y_0), m + n + t = \gcd(x_0 + x_1 + x_2, y_0 + y_1 + y_2), m + t = \gcd(x_0 + x_2, y_0 + y_2)$

Proof: Using theorem 3.1, it is found that $r \mid x$ and $r \mid y$.

Let $z = (z_0, z_1 I_1, z_2 I_2)$ be a common divisor of $x$ and $y$, then
(a) $z_0|x_0, z_0|y_0$, hence $z_0 \mid m$

(b) $z_0 + z_2 \mid x_0 + x_2$ and $z_0 + z_2 \mid y_0 + y_2$, hence $z_0 + z_2 \mid m + t$.

(c) $z_0 + z_1 + z_2 | x_0 + x_1 + x_2$ and $z_0 + z_1 + z_2 | y_0 + y_1 + y_2$, hence $z_0 + z_1 + z_2 \mid m + n + t$.

Thus, applying theorem 3.1, we have $z \mid r$. Hence $r = \gcd(x,y) = (m, nI_1, tI_2)$.

**Example 2.3.4:** Let $x = (3, -I_1, 3I_2), y = (2, 3I_1, I_2)$, then $\gcd(x,y) = (1, -2I_1, 2I_2)$.

**Theorem 2.3.6: (Euclidian division theorem in $Z(I_1, I_2)$ )**

Let $x = (x_0, x_1 I_1, x_2 I_2), y = (y_0, y_1 I_1, y_2 I_2)$ be two elements in $Z(I_1, I_2)$.
There are two corresponding elements $q = (q_0, q_1 I_1, q_2 I_2), r = (r_0, r_1 I_1, r_2 I_2) \in Z(I_1, I_2); x = qy + r$.

**Proof:** By classical division in $Z$, we can find $s_0, p_0, s_1, p_1, s_2, p_2$ such that
$x_0 = y_0 s_0 + p_0, (x_0 + x_2) = s_2(y_0 + y_2) + p_2, (x_0 + x_1 + x_2) = s_1(y_0 + y_1 + y_2) + p_1$.
By putting $q_0 = s_0, q_1 = s_1 - s_2, q_2 = s_2 - s_0, r_0 = p_0, r_1 = p_1 - p_2, r_2 = p_2 - p_0$, we get
$x = qy + r$.

**Example 2.3.5:** Consider $x = (5, 5I_1, -5I_2), y = (3, 2I_1, I_2)$, then we have $q = (1, I_1, -2I_2), r = (2, I_1, 2I_2)$, where $x = qy + r$.

**Remark 2.3.2:** (Solvability of a linear congruence in $Z(I_1, I_2)$)
To solve a linear congruence $x \equiv y(\mathrm{mod}\ z)$. We should take its corresponding equivalent linear congruencies according to Theorem 3.4. Then we can find its solution easily.

**Example 2.3.6:** Consider the following refined neutrosophic linear congruence

$$x \equiv (2, 3I_1, I_2)\big(\mathrm{mod}(1, I_1, 4I_2)\big).$$

The equivalent system of congruencies is

$$x_0 \equiv 2(\mathrm{mod}1) \tag{I}$$

$$x_0 + x_2 \equiv 3(\mathrm{mod}5) \tag{II}$$

$$x_0 + x_1 + x_2 \equiv 6(\mathrm{mod}6) \tag{III}$$

The congruence (I) has a solution $x_0 = 1$. (II) has a solution $x_0 + x_2 = 3$, hence $x_2 = 2$. (III) has a solution $x_0 + x_1 + x_2 = 6$, hence $x_1 = 3$. Thus, the solution of the refined neutrosophic linear congruence is $x = (1, 3I_1, 2I_2)$.

It is easy to check that $(1, I_1, 4I_2) | [(1, 3I_1, 2I_2) - (2, 3I_1, I_2)]$.

**Definition 2.3.5:** Let $Z(I_1, I_2) = \{(a, bI_1, cI_2); a, b, c \in Z\}$ be the refined neutrosophic ring of integers. An element $p = (a, bI_1, cI_2) \in Z(I_1, I_2)$ is said to be prime integer if and only if $p$ is not divided by any other neutrosophic integer different from $(1,0,0)$ and $p$.

**Theorem 2.3.7:** $Z(I_1, I_2)$ has no refined neutrosophic primes.

Proof: Let $p = (a, bI_1, cI_2)$ be any refined neutrosophic integer, different from $(1, 2I_1, -2I_2)$.

We observe that $r = (1, 2I_1, -2I_2)$ is a divisor of p, since $1 | a, 1 - 2 | a + c, 1 + 2 - 2 \mid a + b + c$. This is distinct from both $(1,0,0)$ and $p$. Therefore, $p$ cannot be considered a refined neutrosophic prime.

Additionally, if $p = (1, 2I_1, -2I_2)$, then $(1, -2I_1, 0)$ as a divisor different from p and $(1,0,0)$, thus it can be concluded that refined neutrosophic primes do not exist.

## 3. Comparative Analysis of NRI and RNRI

The key difference between the NRI and RNRI lies in their treatment of the indeterminacy parameter. While the NRI offers a general representation of uncertainty, the RNRI breaks it down into more specific components, providing greater flexibility and precision. However, this comes at the cost of increased computational complexity.

| Aspect | NRI | RNRI |
|---|---|---|
| Indeterminacy Handling | Single indeterminacy parameter | Multiple indeterminacy parameters |
| Computational Complexity | Lower complexity | Higher complexity due to refined parameters |
| Precision | Generalized representation of uncertainty | More precise representation of uncertainty |
| Applications | Useful in general decision theory and cryptography | More suited to advanced AI and engineering problems |

## 4. Conclusion

The neutrosophic ring of integers extends the classical ring theory by incorporating truth, indeterminacy, and falsity, while the refined neutrosophic ring further elaborates on these parameters to handle more nuanced uncertainties. This research explores the structural differences, theoretical properties, and applications of both set up. Conditions for linear Diophantine equations to have solutions are established and attempted to refine the understanding of neutrosophic integer rings by connecting them to classical equations. Additionally, examples are provided to illustrate our approach.

It was observed that the neutrosophic ring of integers (NRI) and the refined neutrosophic ring of integers (RNRI) represent significant advancements in the field of neutrosophic number theory. While both frameworks extend the classical ring of integers to handle uncertainty and indeterminacy, the RNRI offers greater precision by refining the indeterminacy parameter. However, this increased precision comes with added complexity in performing arithmetic operations.

The comparative analysis reveals that while both the NRI and RNRI offer robust ways to model uncertainty, the refined version can provide greater precision in scenarios where higher levels of indeterminacy are prevalent.

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