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ABSTRACT: For two vertices v_j and v_k of a graph G, the usual distance D_{jk}, is the length of the shortest path between vertices v_j and v_k.In this paper, we introduce a new concept named Average Degree D^d Distance energy by considering the average degrees of various vertices presented in the path, in addition to the length of the path for $j \neq k$ and zero otherwise. Also, we compute the Average Degree D^d Distance energy of some standard graphs.

KEYWORDS: Average Degree D^dDistance, Average Degree D^d Distance matrix, Average Degree D^dDistance energy, Cocktail party graph, Crown graph, Friendship graph.

2020 Mathematics Subject Classification: 05C50, 05C12.

1. INTRODUCTION

Throughout the paper, all graphs will be assumed to be simple, connected and finite in order to simplify the analysis. There has been a lot of discussion about the concept of Energy of graph since 1978, when Ivan Gutman introduced it. A more detailed explanation of the concept energy can be found in [2,3].

Let G be a simple undirected graph with vertex set $V'(G) = \{v_1, v_2, \ldots, v_q\}$. The distance D_{jk} between two vertices v_j and v_k is defined as the length of the shortest path connecting v_j and v_k in a graph G . The degree of a vertex u in G, denoted by $d_G(u)$, represents the number of edges connected to u. Recently, T. Jackuline et al[6] defined the concept D^d Distance of graphs. Motivated by this definition, we introduce a new concept called Average Degree D^d Distance energy.

We introduce the Average Degree D^d Distance matrix for a graph G, which is an $q \times q$ matrix $AdD^d(G) = [M_{jk}]$. The elements of this matrix are defined as follows:

$$M_{jk} = \begin{cases} D_{jk} + \left\lfloor \dfrac{d_G(v_j) + d_G(v_k)}{2} \right\rfloor + d_G(v_j)d_G(v_k), & if \;\; j \neq k \\ 0, & if \;\; j = k \end{cases}$$

The eigenvalues of $AdD^d(G)$ labeled as $\mu_1, \mu_2, \dots, \mu_q$ are said to be Average degree D^d Distance eigenvalues of G, and the set of these values is known as the Average degree D^d Distance spectrum of G. The sum of the absolute values of these Average degree D^d Distance eigen values is termed as Average degree D^d Distance energy of G and is represented as $E_{AdD^d}(G)$. Mathematically, this is expressed as:

$$E_{AdD^d}(G) = \sum_{i=1}^{q} |\mu_i|$$

Let I represent the identity matrix. The Average degree D^d Distance polynomial of graph G is defined as follows: $P_{AdD^d}(G,\eta) = \det(\eta I - AdD^d(G))$. Since, the Average degree D^d Distance matrix $AdD^d(G)$ be a real symmetric with zero trace, the sum of eigen values $\mu_i{}'s$ must equal to zero.

2. BASIC CONCEPTS

Lemma 2.1. [4] If u, v, x and y are all matrices with u non singular, then $\begin{vmatrix} u & v \\ x & y \end{vmatrix} = |u||y - xu^{-1}v|$

Lemma 2.2.[4] If u, v, x and y are all matrices. Let $R = \begin{pmatrix} u & v \\ x & y \end{pmatrix}$ if u and x are commutative matrices, then $|R| = |uy - xv|$

Lemma 2.3.[5] The adjacency matrix $A(K_q)$of complete graph K_q, then $A^2(K_q) = (q-2)A(K_q) + (q-1)I_q$.

3. AVERAGE DEGREE D^d DISTANCE ENERGY OF SOME STANDARD GRAPHS

Definition 3.1. The cocktail party graph CP'_{2q} is the graph consisting of two rows of paired vertices in which all vertices but the paired ones are connected with a graph edge.

Theorem 3.2. Let CP'_{2q} be the cocktail party graph. Then

$$S_pAdD^d(CP'_{2q}) = \begin{bmatrix} 8q^3 - 16q^2 + 12q - 2 & -4q^2 + 6q - 2 & -4q^2 + 6q - 4 \\ 1 & q-1 & q \end{bmatrix}$$

and the Average degree D^d distance energy of CP'_{2q} is, $E_{AdD^d}(CP'_{2q}) = 16q^3 - 32q^2 + 24q - 4$ where $q \geq 2$

Proof : Let the cocktail party graph be CP'_{2q} then it has $2q$ vertices, $2q(q-1)$ edges and is $(2q-2)$ – regular.

Let $\alpha = \{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{2q}\}$ be the eigen values of Average degree D^d distance matrix of CP'_{2q}

Then the AdD^d matrix of CP'_{2q} has the form $AdD^d(CP'_{2q}) =$

$$\begin{bmatrix} (4q^2-6q+3)A(K_q) & (4q^2-6q+3)A(K_q)+(4q^2-6q+4)I_q \\ (4q^2-6q+3)A(K_q)+(4q^2-6q+4)I_q & (4q^2-6q+3)A(K_q) \end{bmatrix}$$

The corresponding characteristic polynomial $|\alpha I_{2q} - AdD^d(CP'_{2q})|$ is

$$= \begin{vmatrix} \alpha I_q - (4q^2-6q+3)A(K_q) & -[(4q^2-6q+3)A(K_q)+(4q^2-6q+4)I_q] \\ -[(4q^2-6q+3)A(K_q)+(4q^2-6q+4)I_q] & \alpha I_q - (4q^2-6q+3)A(K_q) \end{vmatrix}$$

$= \left|\left(\alpha I_q - (4q^2-6q+3)A(K_q)\right)^2 - \left((4q^2-6q+3)A(K_q)+(4q^2-6q+4)I_q\right)^2\right|$ by lemma (2.2)

$$= \left|(\alpha^2-[4q^2-6q+4]^2)I_q - 2(4q^2-6q+3)(\alpha+4q^2-6q+4)A(K_q)\right|$$

$$= (2(4q^2-6q+3)(\alpha+4q^2-6q+4))^q \left|\frac{\alpha^2-[4q^2-6q+4]^2}{2(4q^2-6q+3)(\alpha+4q^2-6q+4)} I_q - A(K_q)\right|$$

$$= (2(4q^2-6q+3)(\alpha+4q^2-6q+4))^q \left(\frac{\alpha^2-[4q^2-6q+4]^2}{2(4q^2-6q+3)(\alpha+4q^2-6q+4)} - (q-1)\right)\left(\frac{\alpha^2-[4q^2-6q+4]^2}{2(4q^2-6q+3)(\alpha+4q^2-6q+4)}+1\right)^{q-1}$$

$$=(\alpha+4q^2-6q+4)^q(\alpha+4q^2-6q+2)^{q-1}(\alpha-8q^3+16q^2-12q+2)$$

Hence, the AdD^d spectrum of CP'_{2q} is

$$S_pAdD^d(CP'_{2q}) = \begin{bmatrix} 8q^3-16q^2+12q-2 & -4q^2+6q-2 & -4q^2+6q-4 \\ 1 & q-1 & q \end{bmatrix}$$

and the Average degree D^d distance energy of CP'_{2q} is, $E_{AdD^d}(CP'_{2q}) = 16q^3-32q^2+24q-4$ where $q \geq 2$

Definition 3.3. The crown graph $H'_{q,q}$ is the graph generated from complete bipartite graph by removing perfect matching.

Theorem 3.4. Let $H'_{q,q}$ be the crown graph. Then, spectrum $S_pAdD^d(H'_{q,q})$ is

$$\begin{bmatrix} 2q^3-3q^2+4q & -q^2+2q-4 & -(q^2-q+2)\pm 2 \\ 1 & 1 & each\ (q-1) times \end{bmatrix}$$

and the AdD^d energy is, $E_{AdD^d}(H'_{q,q}) = 2q(2q^2-3q+4)$ where $q \geq 3$

Proof: Let $H'_{q,q}$ be the crown graph then it has $2q$ vertices, $q(q-1)$ edges and is $(q-1)$ - regular.

Let $\eta = \{\eta_1, \eta_2, \eta_3, \dots, \eta_{2q}\}$ be the eigen value. Then the AdD^d matrix of $H'_{q,q}$ is

$$AdD^d\left(H'_{q,q}\right) = \begin{bmatrix} (q^2-q+2)A(K_q) & (q^2-q+3)I_q+(q^2-q+1)A(K_q) \\ (q^2-q+3)I_q+(q^2-q+1)A(K_q) & (q^2-q+2)A(K_q) \end{bmatrix}$$

By lemma (2.3), we get the characteristic polynomial is

$$(\eta-(2q^3-3q^2+4q))(\eta-(-q^2+2q-4))\left(\eta-(-(q^2-q+2)\pm 2)\right)^{q-1}$$

Hence, the AdD^d spectrum $S_pAdD^d(H'_{q,q})$ =

$$\begin{bmatrix} 2q^3-3q^2+4q & -q^2+2q-4 & -(q^2-q+2)\pm 2 \\ 1 & 1 & each\ (q-1)times \end{bmatrix}$$

and the AdD^d energy is, $E_{AdD^d}\left(H'_{q,q}\right) = 2q(2q^2-3q+4)$ where $q \geq 3$

Theorem 3.5. The complete bipartite graph $K_{q,q}$ has the spectrum

$$S_pAdD^d(K_{q,q}) = \begin{bmatrix} -[q^2+q+2] & 2q^3+q^2+2q-2 & -(q^2+2) \\ 2(q-1) & 1 & 1 \end{bmatrix}$$

and the AdD^d energy of $K_{q,q}$ is, $E_{AdD^d}\left(K_{q,q}\right) = 2(2q^3+q^2+2q-2)$ where $q \geq 2$

Proof : Let the complete bipartite graph be $K_{q,q}$ and the eigen value of AdD^d matrix are $\gamma = \{\gamma_1, \gamma_2, \gamma_3, \dots, \gamma_{2q}\}$ respectively. Then the AdD^d matrix of $K_{q,q}$ is

$$AdD^d\left(K_{q,q}\right) = \begin{bmatrix} [q^2+q+2]A(K_q) & (q^2+q+1)[I_q+A\left(K_q\right)] \\ (q^2+q+1)[I_q+A\left(K_q\right)] & [q^2+q+2]A(K_q) \end{bmatrix}$$

By lemma (2.3), we get

$$\left|\gamma I_{2q} - AdD^d(K_{q,q})\right| = (\gamma+[q^2+q+2])^{2(q-1)}\left(\gamma-(2q^3+q^2+2q-2)\right)(\gamma+q^2+2)$$

Therefore,

$$S_pAdD^d(K_{q,q}) = \begin{bmatrix} -[q^2+q+2] & 2q^3+q^2+2q-2 & -(q^2+2) \\ 2(q-1) & 1 & 1 \end{bmatrix}$$

and the AdD^d energy of $K_{q,q}$ is, $E_{AdD^d}\left(K_{q,q}\right) = 2(2q^3+q^2+2q-2)$ where $q \geq 2$

Theorem 3.6. Let the complete graph be K_q. Then

$$S_pAdD^d(K_q) = \begin{bmatrix} -[q(q-1)+1] & q^3-2q^2+2q-1 \\ q-1 & 1 \end{bmatrix}$$

and the AdD^d energy of K_q is, $E_{AdD^d}\left(K_q\right) = 2(q^3-2q^2+2q-1)$ where $q \geq 2$

Proof : Let the eigen value of complete graph K_q be $\beta' = \{\beta'_1, \beta'_2, \beta'_3, \dots, \beta'_q\}$. Then it has q vertices, $\frac{q(q-1)}{2}$ edges and is $(q-1)$ - regular.

Then the AdD^d matrix of K_q has the form

$$AdD^d(K_q) = [(q(q-1)+1)A(K_q)]$$

By lemma (2.3), we get

$$(\beta' + [q(q-1)+1])^{q-1}(\beta' - (q^3 - 2q^2 + 2q - 1))$$

Hence,

$$S_pAdD^d(K_q) = \begin{bmatrix} -[q(q-1)+1] & q^3 - 2q^2 + 2q - 1 \\ q-1 & 1 \end{bmatrix}$$

and the AdD^d energy of K_q is, $E_{AdD^d}(K_q) = 2(q^3 - 2q^2 + 2q - 1)$ where $q \geq 2$

Theorem 3.7. For any $q \geq 1$, the star graph $K_{1,q}$ has the spectrum

$$S_pAdD^d(K_{1,q}) = \begin{bmatrix} -4 & 2(q-1) \pm \sqrt{4(q-1)^2 + q\left\lfloor \frac{3(q+1)}{2} \right\rfloor^2} \\ q-1 & each\ 1\ time \end{bmatrix}$$

and the AdD^d energy of $K_{1,q}$ is, $E_{AdD^d}(K_{1,q}) = 4(q-1) + 2\sqrt{4(q-1)^2 + q\left\lfloor \frac{3(q+1)}{2} \right\rfloor^2}$

Proof : Let us take a star graph $K_{1,q}$ with vertex set $V'(K_{1,q}) = \{v'_0, v'_1, v'_2, \dots, v'_q\}$ where the vertex v'_0 has degree q and the eigen values of AdD^d matrix are $\mu = \{\mu_0, \mu_1, \dots, \mu_q\}$ respectively.

For this graph, the AdD^d matrix is

$$AdD^d(K_{1,q}) = \begin{bmatrix} 0 & \left\lfloor\frac{3(q+1)}{2}\right\rfloor & \left\lfloor\frac{3(q+1)}{2}\right\rfloor & \left\lfloor\frac{3(q+1)}{2}\right\rfloor & \cdots & \cdots & \left\lfloor\frac{3(q+1)}{2}\right\rfloor & \left\lfloor\frac{3(q+1)}{2}\right\rfloor \\ \left\lfloor\frac{3(q+1)}{2}\right\rfloor & 0 & 4 & 4 & \cdots & \cdots & 4 & 4 \\ \left\lfloor\frac{3(q+1)}{2}\right\rfloor & 4 & 0 & 4 & \cdots & \cdots & 4 & 4 \\ \left\lfloor\frac{3(q+1)}{2}\right\rfloor & 4 & 4 & 0 & \cdots & \cdots & 4 & 4 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \left\lfloor\frac{3(q+1)}{2}\right\rfloor & 4 & 4 & 4 & \cdots & \cdots & 0 & 4 \\ \left\lfloor\frac{3(q+1)}{2}\right\rfloor & 4 & 4 & 4 & \cdots & \cdots & 4 & 0 \end{bmatrix}$$

On simplification, we get the characteristic polynomial

$$(\mu+4)^{q-1}\left(\mu^2 - 4\mu(q-1) - q\left\lfloor\frac{3(q+1)}{2}\right\rfloor^2\right)$$

and the spectrum would be

$$S_pAdD^d(K_{1,q}) = \begin{bmatrix} -4 & 2(q-1) \pm \sqrt{4(q-1)^2 + q\left\lfloor\frac{3(q+1)}{2}\right\rfloor^2} \\ q-1 & each\ 1\ time \end{bmatrix}$$

and the AdD^d energy of $K_{1,q}$ is, $E_{AdD^d}(K_{1,q}) = 4(q-1) + 2\sqrt{4(q-1)^2 + q\left\lfloor\frac{3(q+1)}{2}\right\rfloor^2}$

Definition 3.8. The friendship graph, denoted by F_q^3, is defined as the graph obtained by taking q copies of the cycle graph C_3 with a vertex in common.

It is clear that $\left|V\left(F_q^3\right)\right| = 2q+1$ and $\left|E\left(F_q^3\right)\right| = 3q$

Theorem 3.9.The spectrum and energy of the friendship graph F_q^3 is

$$S_pAdD^d\left(F_q^3\right) = \begin{bmatrix} -9 & -7 & \frac{16q-9 \pm \sqrt{200q^3+416q^2-256q+81}}{2} \\ q-1 & q & each\ 1\ time \end{bmatrix}$$

and$E_{AdD^d}\left(F_q^3\right) = 16q - 9 + \sqrt{200q^3+416q^2-256q+81}$

Proof: Consider the friendship graph F_q^3 with vertices set $V'(F_q^3) = \{v'_0, v'_1, v'_2, \dots, v'_{2q}\}$ where $\deg(v'_0) = 2q$ and the eigen values of AdD^d matrix are $\eta' = \{\eta_0, \eta_1, \dots, \eta_{2q}\}$ respectively.

Then the AdD^d matrix of F_q^3 is

$$AdD^d(F_q^3) = \begin{bmatrix} 0 & 2(2q+1)+q & 2(2q+1)+q & \cdots & \cdots & \cdots & 2(2q+1)+q & 2(2q+1)+q \\ 2(2q+1)+q & 0 & 7 & \cdots & \cdots & \cdots & 8 & 8 \\ 2(2q+1)+q & 7 & 0 & \cdots & \cdots & \cdots & 8 & 8 \\ 2(2q+1)+q & 8 & 8 & \cdots & \cdots & \cdots & 8 & 8 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2(2q+1)+q & 8 & 8 & \cdots & \cdots & \cdots & 0 & 7 \\ 2(2q+1)+q & 8 & 8 & \cdots & \cdots & \cdots & 7 & 0 \end{bmatrix}$$

The characteristic polynomial becomes

$$(\eta+9)^{q-1}(\eta+7)^q(\eta^2-(16q-9)\eta-2q(5q+2)^2)$$

$$(\eta+9)^{q-1}(\eta+7)^q\left(\eta-\left(\frac{16q-9\pm\sqrt{200q^3+416q^2-256q+81}}{2}\right)\right)$$

Hence, the spectrum

$$S_pAdD^d(F_q^3) = \begin{bmatrix} -9 & -7 & \frac{16q-9\pm\sqrt{200q^3+416q^2-256q+81}}{2} \\ q-1 & q & each\ 1\ time \end{bmatrix}$$

and $E_{AdD^d}(F_q^3) = 16q-9+\sqrt{200q^3+416q^2-256q+81}$

4. CONCLUSION

Throughout the various fields of chemistry, biology and computer science, there are numerous applications of Average Degree D^d Distance energy of a graph. Thus, the study of Average Degree D^d Distance energy remains an active and productive area of research, with implications that go far beyond the domain of graph theory in terms of its applications. In future, we extend this Average degree D^d Distance energy to many graphs.

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