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Abstract: *Drug addiction is a serious public health epidemic with far-reaching societal, economic, and personal consequences. This paper proposes an optimal control policy for the deterministic drug addiction model. The model takes into account three state variables: suspectables (those at risk of addiction), infected (drug addicts), and recovered. The model also accounts for factors such as drug addicts' deaths, natural recovery from drug addiction, and so on. The proposed drug addiction model's optimal control analysis is carried out utilizing Pontryagin's maximum principal. The prerequisites for optimal control of the drug addiction problem are derived and analyzed using effective drug addiction control approaches.*

Keywords: *Drug addiction, optimal control, mathematical model, Pontryagin's maximum principal.*

1. Introduction

Drug addiction is a serious public health epidemic with far-reaching societal, economic, and personal consequences. Addiction has far-reaching consequences for families, communities, and healthcare systems. Mathematical modelling has proven to be an effective technique for understanding the dynamics of infectious diseases, and similar models can be used to study the spread of drug addiction in a population. One such technique is the SIR (Susceptible-Infected-Recovered) model, which is widely used in epidemiology and can be used to investigate addiction as a transmissible condition impacted by social and environmental factors.

In this regard, the SIR model divides people into three groups: susceptible (those at risk of addiction), infected (those who are already addicted), and recovered. Transition rates between these compartments are determined by factors such as peer influence, therapeutic efforts, and recovery rates. However, without targeted intervention, the addiction crisis can spread uncontrolled, resulting in higher rates of addiction and associated socioeconomic costs.

Optimal control theory provides a framework for developing and implementing techniques that reduce addiction rates and the costs associated with prevention and treatment. The SIR model can be modified to analyse the effects of various intervention techniques over time by integrating control variables, such as preventative programs to lower susceptibility and treatment programs to improve recovery. Through optimal control analysis, policymakers can identify the most effective and cost-efficient approaches for addressing drug addiction within a given population.

This paper gives an optimal control study of a SIR model modified for drug addiction, examining the impact of preventative and treatment strategies. By establishing a cost functional that includes addiction prevalence and intervention costs, we can construct and solve an optimum control problem that reduces addiction's overall impact. The findings emphasise the potential of targeted interventions to reduce addiction rates and direct resource allocation for maximum societal benefit. The work is inspired by [1], which suggested and analysed a non-linear optimum control model for drug usage. The basic SIR model for drug addiction and other social issues [2] is discussed. The study examines the optimal control strategy applied to a dynamic model of drug abuse incidents in order to reduce its adverse effects [3]. The author [4] proposed and discussed Everingham and Rydell's first-order difference equation model of cocaine use, which is used to formulate and solve preventing and treatment spending decisions in the farmwork of dynamic optimal control under various assumptions about how freely drug control budgets can be changed. The optimal control analysis of a mathematical model for unemployment [5] is discussed.

2. Mathematical model for drug addiction

The optimal control problem is solved using Pontryagin's Maximum Principle, a mathematical tool that specifies the criteria for best performance. Using this idea, we obtain a system of differential equations known as the optimality system, which describes the dynamics of both the population compartments and the costate variables linked with the controlling variables. Numerical approaches, such as gradient-based optimisation methods, are used to solve these equations and determine the optimal values for $u_1(t)$ and $u_2(t)$ over time.

The study is performed over a predetermined time period, during which the control variables $u_1(t)$ and $u_2(t)$ are changed to strike the best balance between reducing addiction prevalence and intervention costs. Numerical simulations are then performed to see how these controls affect population dynamics, comparing situations with and without optimal control.

Mathematical Model for drug addiction without control

$$\frac{ds}{dt} = -\beta SI - S$$

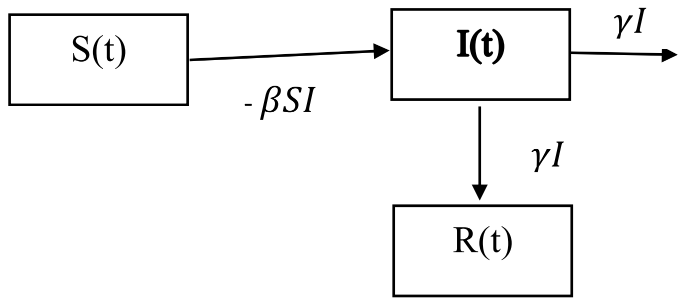
$$\frac{dI}{dt} = \beta SI - \gamma I - I \qquad (1)$$

$$\frac{dR}{dt} = \gamma I + I$$

Table 1: Variable and Parameters

Variables/Parameters	Explanation
S	Individuals at risk of becoming addicted.
I	Individuals who are addicted.
R	Individuals who have recovered from addiction and are not currently using drugs.
β	The transmission rate of addiction.
γ	The natural recovery rate from addiction.

Figure 1: Model flow diagram



2.1. Mathematical model for drug addiction with control

To develop an effective control method, we first structure the dynamics of drug addiction using the SIR model framework. Individuals in the Susceptible compartment (S) are people who are predisposed to addiction due to variables such as peer pressure, socioeconomic status, or personal situations. When vulnerable individuals encounter addicted persons, they may enter the Infected compartment (I), which represents those who are actively dealing with addiction. Finally, with appropriate intervention or personal efforts, individuals can go from the Infected to the Recovered compartment (R), which represents those who have overcome addiction and, ideally, are at a low risk of recurrence. In the context of drug addiction, prevention and treatment are critical measures. Education, community outreach, and support programs aimed at reducing addiction vulnerability are all possible prevention initiatives. Counselling, rehabilitation, and medical support are examples of treatment procedures that help persons who are currently addicted recover. However, both prevention and treatment are costly, mandating the most efficient use of scarce resources.

To balance these considerations, we introduce two control variables:

1. Prevention Control $u_1(t)$ – Reduces the transition rate from Susceptible to Infected, effectively lowering the risk of addiction initiation.
2. Treatment Control $u_2(t)$ – Increases the transition rate from Infected to Recovered, supporting the recovery of those already struggling with addiction.

$$\frac{ds}{dt} = -\beta SI - u_1(t)S$$

$$\frac{dI}{dt} = \beta SI - \gamma I - u_2(t)I \qquad (2)$$

$$\frac{dR}{dt} = \gamma I + u_2(t)I$$

The optimal control analysis aims to reduce the overall social cost of drug addiction, which includes both the prevalence of addiction and the financial costs of prevention and treatment treatments. This goal can be expressed as a cost functional that we aim to minimise over a given time horizon. The cost functional consists of three main terms: Addiction Prevalence: This word refers to the impact of addiction on society as a function of the number of actively addicted persons (I) throughout time.

Prevention Cost: This phrase refers to the expenses associated with executing prevention initiatives, as represented by $u_1(t)$ and its weight.

Treatment Cost: This term reflects the expenses of giving treatment to addicted people, as represented by $u_2(t)$ and its related weight. By quantifying and integrating these costs throughout time, we may create an objective function that captures the overall burden of addiction and therapies. The optimal control problem thus consists of determining the values of $u_1(t)$ and $u_2(t)$ that minimise this objective function while sticking to the dynamics of the SIR model.

$$J = \int_0^T (AI + Bu_1(t)^2 + Cu_2(t)^2)\mathrm{dt} \tag{3}$$

'A' is the weight for addiction prevalence, 'B' and 'C' are weights for the costs associated with control interventions $u_1(t)$ and $u_2(t)$, respectively.

To construct the Hamiltonian H, we introduce adjoint (costate) variables λ_S, λ_I and λ_R associated with the state variables S, I, and R, respectively. The Hamiltonian is then defined as:

$$H = AI + Bu_1(t)^2 + Cu_2(t)^2 + \lambda_S(-\beta SI - u_1(t)S) + \lambda_I(\beta SI - \gamma I - u_2(t)I) + \lambda_R(\gamma I + u_2(t)I) \tag{4}$$

AI: Cost due to addiction prevalence, $Bu_1(t)^2$ and $Cu_2(t)^2$: Control costs for prevention and treatment, $\lambda_S(-\beta SI - u_1(t)S)$: Contribution from the Susceptible equation, $\lambda_I(\beta SI - \gamma I - u_2(t)I)$: Contribution from the Infected equation and $\lambda_R(\gamma I + u_2(t)I)$: Contribution from the Recovered equation.

Using the Hamiltonian, the adjoint equations are given by:

$$\frac{d\lambda_S}{dt} = \frac{\partial H}{\partial s}$$

$$\frac{d\lambda_I}{dt} = \frac{\partial H}{\partial I} \tag{5}$$

$$\frac{d\lambda_R}{dt} = \frac{\partial H}{\partial R}$$

Computing these partial derivatives will yield the system of differential equations for the adjoint variables λ_S, λ_I and λ_R.

The optimal controls $u_1(t)$ and $u_2(t)$ are obtained by setting:

$$\frac{\partial H}{\partial u_1(t)} = 0 \quad \text{and} \quad \frac{\partial H}{\partial 2(t)} = 0$$

This leads to:

$$u_1(t) = min\left(max\left(0, -\frac{\lambda_S S}{2B}\right), u_1(t), max\ \right)$$

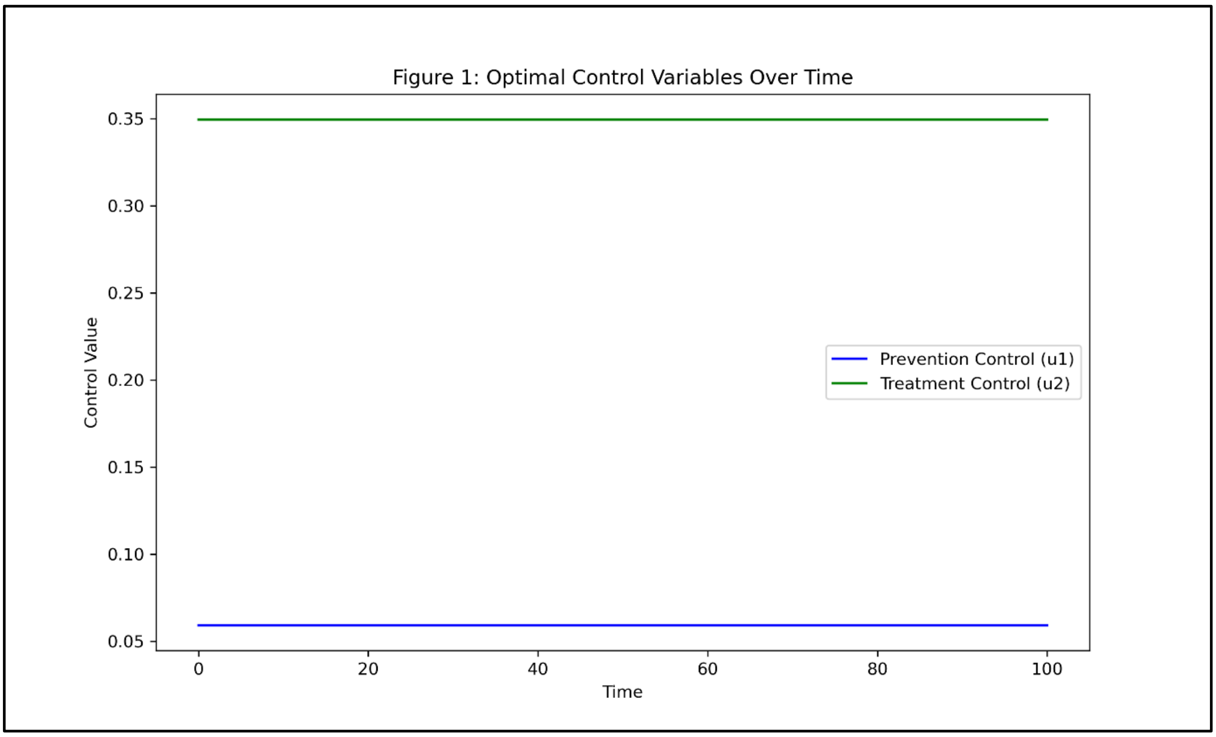
$$u_2(t) = min\left(max\left(0, -\frac{\lambda_I I}{2C}\right), u_2(t), max\ \right)$$

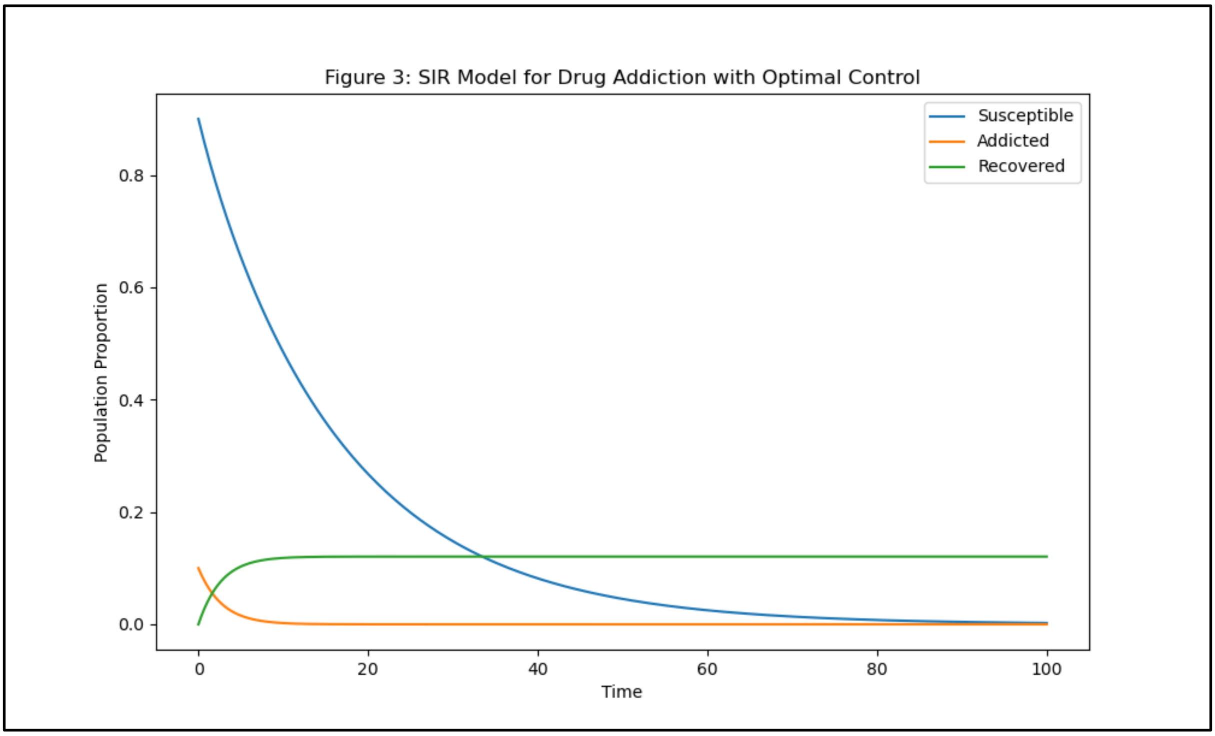
Where $u_1(t), Max$ and $u_2(t), max$ are the upper bounds for the controls.

3. Numerical simulation

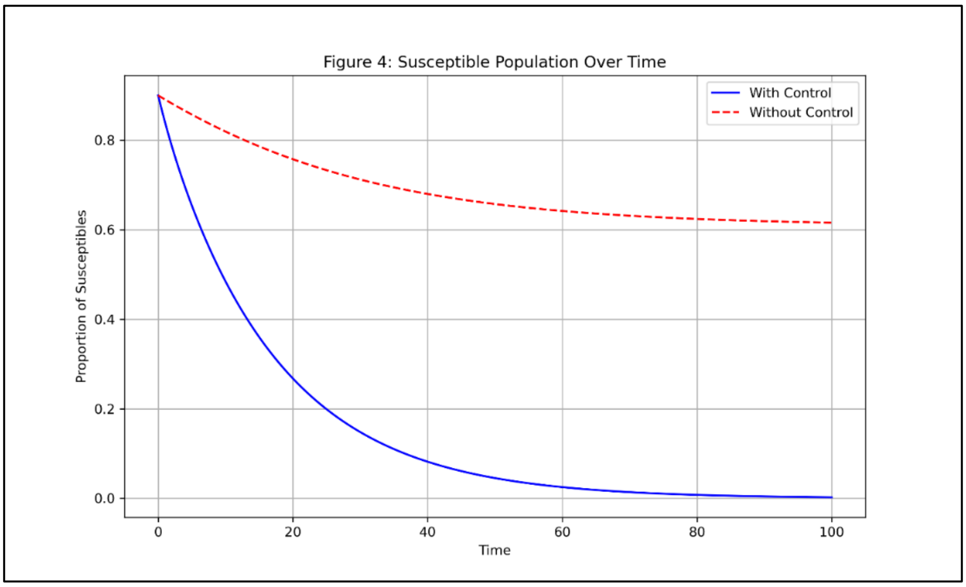
This section discusses the numerical simulations of the optimality system and the corresponding results of varying the optimal controls $u_1(t)\ and\ \ u_2(t)$ and some parameter choices and interpretation from various cases using base line parameter values. Numerical solutions to the optimality system composing the state eqn. (2) and adjoint equations (5) are carried out in python 3.0 using parameters $\beta = 0.1\ and\ \gamma = 0.1$ together with the following weight $A = 1, B = 0.01\ and\ C = 0.01$ and initial conditions, $S(0) = 0.9, I(0) = 0.1\ and\ R(0) = 0.$

The blue line represents the optimal level of the prevention control over time. Since this value is constant, it suggests that a steady level of prevention effort is optimal for reducing the number of individuals transitioning from susceptible to addicted over the entire period. The green line shows the optimal level of the treatment control, which also remains constant over time. This consistent treatment level implies that maintaining a steady recovery rate helps manage addiction prevalence effectively (Figure 2).

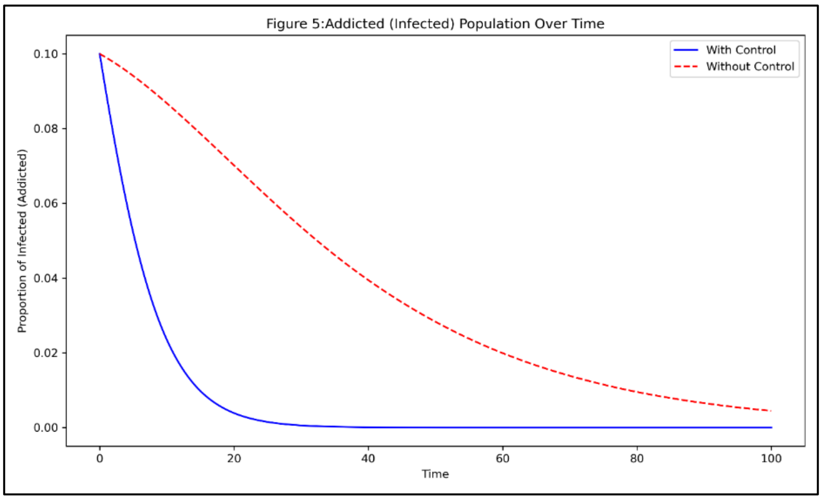




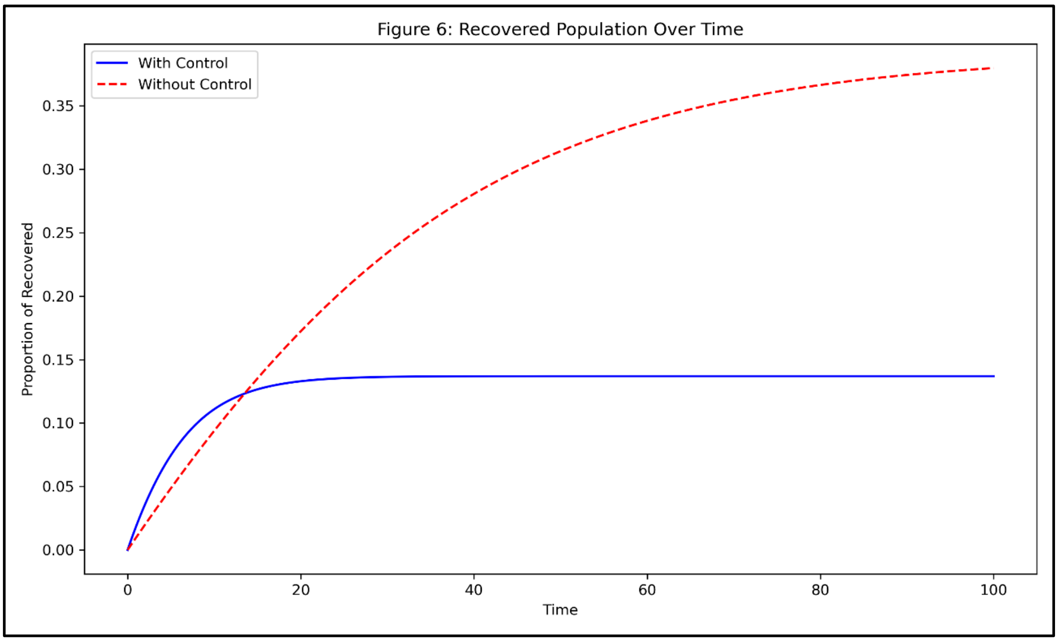
The graphic depicts the dynamics of the SIR model for drug addiction under optimal management measures. The figure depicts the efficacy of the best control method in managing addiction in the population. Prevention initiatives reduce the number of new addiction cases, and therapy accelerates recovery. These strategies work together to lower addiction prevalence and promote a better balance among susceptible, addicted, and recovered populations (Figure 3).



Plots comparing the outcomes of the SIR model under two scenarios: with optimal control and without control. Susceptible Population Over Time. The blue line reflects the fraction of susceptible individuals when optimal control is implemented, whereas the red dashed line depicts the susceptible population in the absence of any control actions. With control, the susceptible population declines more slowly, indicating that the preventative technique (u_1 (t)) is effective in slowing the rate at which people get hooked. Without control, the susceptible population declines more quickly, since more people get addicted owing to a lack of prevention (Figure 4).



Plots comparing the outcomes of the SIR model under two scenarios: with optimal control and without control. Addicted Population Over Time: The blue line depicts the fraction of addicted people with optimal control, whereas the red dashed line indicates the addicted population without control. Under management, the addicted population rises initially but peaks at a lower level before declining as a result of both prevention and treatment efforts $u_1(t)$ and $u_2(t)$. This indicates the efficacy of control strategies in reducing addiction prevalence. Without control, the addicted population peaks higher and stays there for longer, as there are no treatments to curb addiction spread or improve recovery (Figure 5).



Plots comparing the outcomes of the SIR model under two scenarios: with optimal control and without control. The population has gradually recovered. The blue line reflects the recovered population under control, while the red dashed line depicts recovery without control. With control, recovery occurs more quickly and in a greater proportion as a result of the treatment intervention $u_2(t)$, allowing more people to move from addiction to recovery. Without control, recovery is delayed, and fewer people eventually recover, demonstrating the need of treatment in boosting overall recovery rates (Figure 6).

4. Conclusion:

In this work, an optimal control analysis for a drug addiction model is performed using Pontryagin's Maximum Principle. The requirements for optimal control of a drug addiction problem are derived and assessed with the effective use of control variables through prevention and treatment. It is determined that a successful prevention policy has a considerable impact on drug addiction reduction. Control initiatives based on these tactics can effectively lessen the drug addiction problem in society.

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