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ESTIMATING COSMOLOGICAL MODEL PARAMETERS USING TYPE IA SUPERNOVAE DATA

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Abstract: In this work, the author uses 13 magnitude-redshift data of a Type Ia supernovae, from Ratra et al. (2011) [1]. to estimate cosmological model parameters of the λCDM model. Author considers a spatially flat two-component model universe to estimate the matter density Ω_{m0} and the cosmological constant $\Omega_{\lambda 0}$, of the Friedmann equations, applying the maximum likelihood test. Using simple statistical methods constraint contours are plotted around the best fit value in the $\Omega_{m0} - \Omega_{\lambda 0}$ plane. The mean values of the Hubble parameter for all the 13 redshifts are computed using Monte Carlo simulation. The results obtained are in good agreement with the observed values.

Keywords: Cosmological model parameter, Hubble Constant, Redshift, Spacetime

1. Introduction

Cosmologists try to model the universe in terms of parameters. The values of these parameters can be estimated theoretically and checked whether it fits the observational data. In 1929, Edwin Hubble discovered that the universe is expanding, by measuring the recessional velocities of distant galaxies. He observed that the galaxies that are further away are receding faster than the galaxies that are close to our galaxy. Hubble constant is the slope of the relation between the distance to recessional velocity of galaxies. The method adopted by Hubble was to measure the redshifts of the spectral lines emitted from distant galaxy and using this to calculate the recessional velocity of the galaxies. What he obtained is basically a linear relation between the recessional velocity and the distance. By now it has been well established that the universe is going through phase of accelerated expansion. Though the exact cause of this expansion is not known, there are plethora of cosmological models that try explain the observed expansion. This work considers a two-component flat universe model and try to estimate two model parameters using basic statistical methods.

2. Theoretical Background

Cosmological principle is the hypothesis that the universe is spatially homogeneous and isotropic. The term homogeneity implies that the physical quantity measured by one is the same at two points *P* and *Q* in *spacetime*. Isotropy at a given point *P* in *spacetime* implies invariance with respect to change in direction at *P*. In mathematical terms, we represent the fact by the following equations.

$$g_{\mu\nu}(y) = g'_{\mu\nu}(y) \tag{1}$$

$$T_{\mu\nu}(y) = T'_{\mu\nu}(y) \tag{2}$$

Where $g_{\mu\nu}$ is the metric tensor and $T_{\mu\nu}$ is the energy-momentum tensor. The first equation represents the fact that the coordinate transformation is an isometry and the second equation implies the form invariance.
Redshift is a measure of how much the observed wavelength of light differs from that of the emitted. The redshift *z* of light emitted from galaxies is given by

$$z = \frac{\lambda}{\lambda_o} - 1 \tag{3}$$

where λ_o is the wavelength of the spectral line emitted from a galaxy and λ is the observed wavelength. In a constant gravitational field, the redshift may be treated to be corresponding to a recession velocity v. For objects that are nearby, the velocity v can be predicted by simple Doppler formula

$$v = cz \tag{4}$$

2.1 Elements of FRW cosmology

The dynamics of the universe is described by the Einstein equations which are in general complicated nonlinear equations. Simple analytic solutions are possible in the presence of generic symmetries. The Friedmann-Robertson-Walker (FRW) metric is based upon the assumption of homogeneity and isotropy of the universe which can be treated approximately true on large scales. The FRW metric in its most general form reads

$$ds^2 = -cdt^2 + a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta \, d\phi^2\right] \tag{5}$$

where a(t) is the scale factor with cosmic time t. The coordinates r, θ and ϕ are known as comoving coordinates. The constant *k* describes the geometry of the spatial section of the *spacetime*. Hubble's law states that the recessional velocity of a galaxy moving away from Earth is proportional to its distance.

$$v(t) = H(t)d(t) \tag{6}$$

Where v(t) is the recessional velocity, H(t) is the Hubble parameter and d(t) is the proper distance.

$$H(t) = \frac{\dot{a(t)}}{a(t)} \tag{7}$$

The value of the Hubble parameter $H(t)$ at the present epoch is the Hubble constant H_o. The overall geometry of the universe is determined by the density of matter in it. The density parameter Ω is the ratio of the actual or the observed density ρ to the critical density ρ_c of the universe. The Friedmann equation seen in terms of the density parameters, reads

$$\frac{H(t)^2}{H_o^2} = \sqrt{\Omega_{r,0}\, a^{-4} + \Omega_{m,0}a^{-3} + \Omega_{\lambda,0} + (1 - \Omega_0)a^{-2}} \tag{8}$$

Where $\Omega_{r,0}$ is the radiation density at the present epoch, $\Omega_{m,0}$ is the matter (baryonic plus dark) density at the present epoch, $(1 - \Omega_0)$ is the spatial curvature density at the present epoch and $\Omega_{\lambda,0}$ is the cosmological constant which is interpreted to be the vacuum energy density at the present epoch. For the purpose of parameter estimation, a two-component model of the universe is considered by assuming, a spatially flat universe ($k = 0$) and radiation density to be zero.

$$\frac{H(t)^2}{H_o^2} = \sqrt{\Omega_{m,0}a^{-3} + \Omega_{\lambda,0}} \tag{9}$$

The evolution of the universe is often described using redshift. This is due to the fact that light emitted from a stellar object becomes red-shifted due to the expansion of the universe. The wavelength λ increases proportionally to the scale factor a(t), whose effect can be quantified by the redshift z, as

$$1 + z = \frac{\lambda_0}{\lambda} = \frac{a_0}{a} \tag{10}$$

The Hubble parameter in terms of the redshift z reads

$$H(z) = H_o\sqrt{(1 + z)^3\Omega_{m,0} + \Omega_{\lambda,0}} \tag{11}$$

3. Statistical Methods

The Hubble parameter versus redshift data of a Type-Ia supernovae is statistically analyzed with three statistical tools viz, χ^2-test, Likelihood estimate and Monte Carlo simulation to provide an estimate of the two model parameters; the matter density $\Omega_{m,0}$ and the Cosmological constant $\Omega_{\lambda,0}$. A χ^2-test is done obtain a best fit value of the model parameters by minimizing χ^2 and obtaining χ^2_{min}. The difference $\Delta\chi^2 = \chi^2_{min} - \chi^2$

can then be used to plot constraint contours for the pair of model parameters $(\Omega_{m0}, \Omega_{\lambda 0})$. A Likelihood function provide good estimate of the parameters. Plot of one-dimensional likelihood function corresponding to each of the model parameters by means of marginalization provides an estimate of the values of the density parameters and the Hubble constant. A Monte Carlo simulation is performed to estimate Hubble Parameters corresponding to each redshift z and compute the mean and the standard deviation of the results obtained from the simulation.

3.1 Chi-Squared Test

First step in finding whether an observational data fits the theory is to check whether the theory is consistent with the data of its viable values of free parameters. One way of doing this is the maximum likelihood approach. The χ^2 reads

$$\chi^2 = \sum_{j=1}^{N} \frac{[m_t(z_j; \text{parameters}) - m_{0,j}]^2}{\sigma_{mj}^2} \tag{12}$$

where $m_t(z_j; \text{parameters})$ is the theoretical value of magnitude m_t corresponding to a redshift z_j of the jth supernova predicted by a model which is given in terms of its parameters [4]. The variance σ_{mj}^2 represents the combined uncertainty in the observed magnitude of the jth supernova due to the uncertainties in different variables. The χ^2 for estimating the parameters of a two-component flat universe model using the Hubble parameter $H(z)$ reads

$$\chi^2 = \sum_{j=1}^{N} \frac{[H_0\sqrt{(1+z)^3\Omega_{m,0}+\Omega_{\lambda,0}}-H(z_i)^{obs}]^2}{\sigma_H^2} \tag{13}$$

The 13 data sets of redshifts, z with corresponding $H(z)$ and σ_H is used to plot χ^2 as a function of the two model parameters $\Omega_{m,0}$ and $\Omega_{\lambda,0}$, and hence find the pair of values $(\Omega_{m0}, \Omega_{\lambda 0})$ for which the value o χ^2 is minimum. Table 1. shows the Hubble parameter versus redshift data of 13 supernovae [4].

Table 1. Hubble Parameter vs Redshift Data

| z	$H(z)$	σ_H
0.1	69	12
0.17	83	8
0.24	79.69	2.65
0.27	77	14
0.4	95	17
0.43	86.45	3.68
0.48	97	60
0.88	90	40
0.9	117	23
1.3	168	17
1.43	177	18
1.53	140	14
1.75	202	40

Source: Ratra *et al* [4].

Range of the model parameters: $0.1 < \Omega_{m,0} < 1, \ 0.1 < \Omega_{\lambda,0} < 1$. χ^2 is calculated using data from Table 1 and a surface plot of χ^2 versus the model parameters $\Omega_{m,0}$ and $\Omega_{\lambda,0}$ is shown in figure 1.
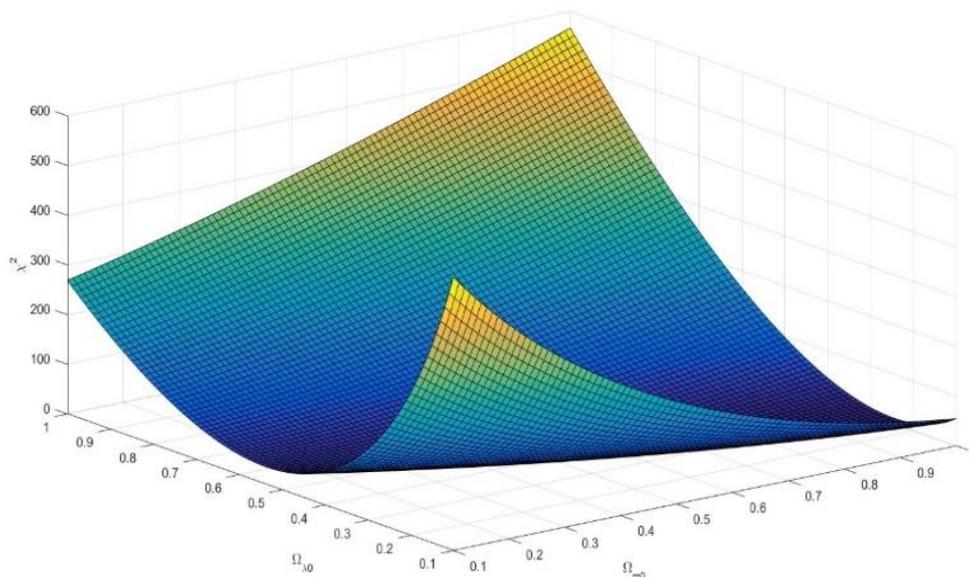
Figure 1. Surface Plot of χ^2 versus $\Omega_{m,0}$ and $\Omega_{\lambda,0}$

The χ^2 has a minimum value for certain values of the model parameters $\Omega_{m,0}$ and $\Omega_{\lambda,0}$, and then it shoots up to higher ranges for certain other values of the parameters. There is a range of values of the model parameters for which the χ^2 maintains a value not far from the minima. For the 13 data sets of Table 1. the minimum χ^2 is 8.1015 which corresponds to $\Omega_{m,0} = 0.3100, \Omega_{\lambda,0} = 0.780$ which in turn are the best fit values.
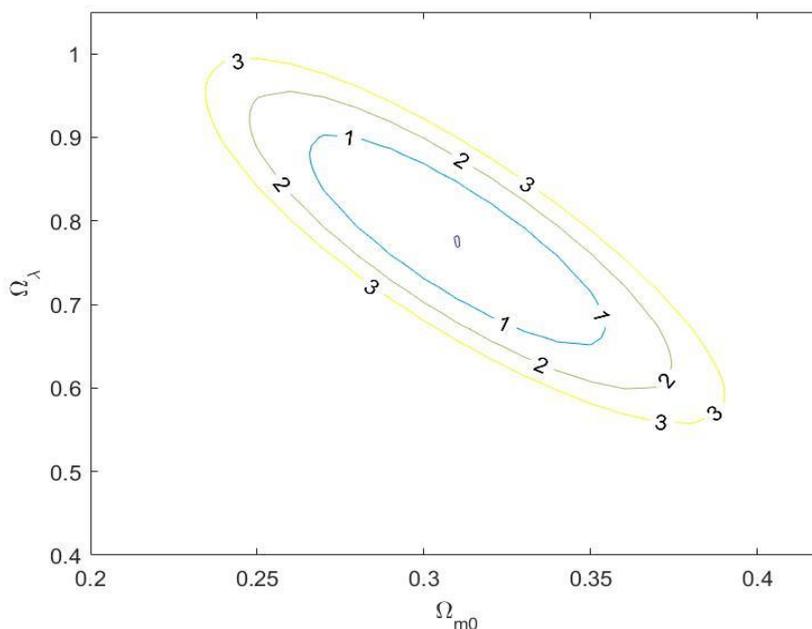
Figure 2. $\Delta\chi^2 = 1, 2$ and 3 constraint contours around the best-fit pair with $(\Omega_{m0}, \Omega_{\lambda 0}) = (0.31, 0.780)$.

3.2. Likelihood Estimation

One-dimensional likelihoods for models having two or more parameters can be obtained by marginalizing over the parameters which are not of interest, to get a likelihood function of reduced dimension. For a Gaussian probability distribution function, the likelihood function for a two-component flat universe model reads

$$L(\Omega_{m0}, \Omega_{\lambda0}) = e^{-\chi^2(H_o, \ \Omega_{m0}, \ \Omega_{\lambda0})/2}$$

Marginalization is done by integrating the likelihood function over those parameters on which we are not interested which are called the nuisance parameter. Doing this yields a likelihood function that depends on the parameters of interest only. For a two-component universe we can make use of the fact that $\Omega_{m0} = 1 - \Omega_{\lambda0}$ and reduce the likelihood function as a function of two parameters. Two possible choices of the likelihood function are $L(H_o, \Omega_{m0})$ and $L(H_o, \Omega_{\lambda0})$. In each case the likelihood function is marginalized over one of the arguments.

$$L(\Omega_{m0}) = \int L(H_o, \Omega_{m0})dH_o$$

$$L(\Omega_{\lambda0}) = \int L(H_o, \Omega_{\lambda0})dH_o$$

Marginalization is done on H_o with a flat prior over the range 50 to 85 $kms^{-1}Mpc^{-1}$ to get a likelihood as a function of one variable parameter [3]. The marginalization ranges and the best fit values are listed in Table 2. Best fit values are the values which yield thw maximum likelihood.

Table 2. Marginalization ranges and the best fit values obtained.

Likelihood Function	Marginalization Range	Best-fit value
$L(H_o, \Omega_{m0})$	$50 < H_o < 85$	$\Omega_{m0} = 0.28$
$L(H_o, \Omega_{\lambda0})$	$50 < H_o < 85$	$\Omega_{\lambda0} = 0.72$
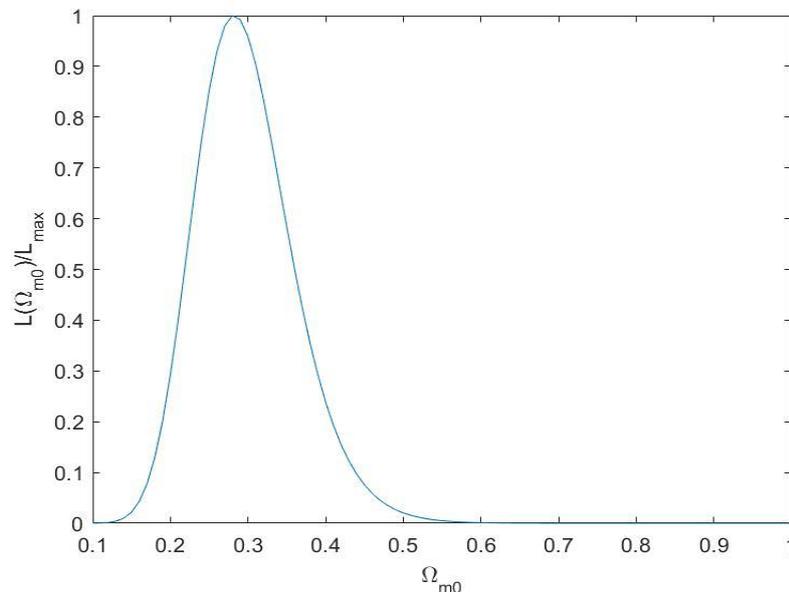
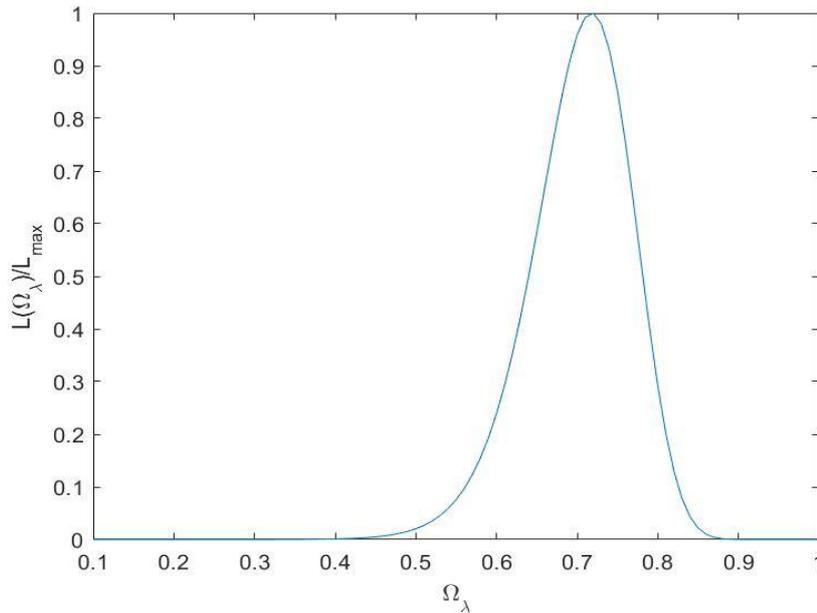**Figure 3. Plot of $L(H_o, \Omega_{m0})$ versus Ω_{m0}**

Figure 4. Plot of $L(H_o, \Omega_{\lambda 0})$ versus $\Omega_{\lambda 0}$
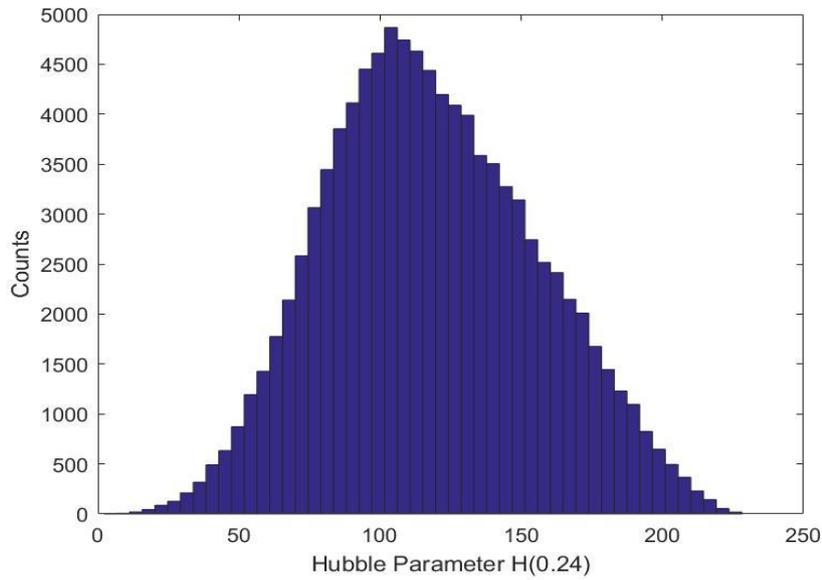
3.3. Monte Carlo Simulation

Monte Carlo methods use sequences of random numbers to solve problems. Random numbers serve as an important tool in simulating stochastic process where a variable varies randomly in time. This technique can be used to sample random variables governed by complicated probability density functions. A Monte Carlo simulation can be performed to estimate the Hubble Parameter H(z) for the dataset of table 1. by randomizing the three model parameters viz, H_o, Ω_{m0}, and $\Omega_{\lambda 0}$ with certain probability density function and than study the corresponding distribution in the values of Hubble Parameter.

First set of results are obtained from uniform distribution of all three model parameters in the ranges specified in table 3. The three model parameters are randomized in the specified ranges and the results are obtained for 100000 iterations. The Histogram plot of H(z) shows Gaussian behavior. The mean values of H(z) and the standard deviation σ_H, corresponding to each of the 13 redshifts z, obtained from the Monte Carlo simulation is listed in Table 4. The Hubble constant H_o is considered uniformly distributed in the 1σ range i.e., $68 \pm 2.8 \text{ km}^{-1}\text{Mpc}^{-1}$ [2]. The second set of results are obtained taking H_o to be normally distributed with mean, $68 \text{ km}^{-1}\text{Mpc}^{-1}$ and standard deviation $2.8 \text{ km}^{-1}\text{Mpc}^{-1}$. The distribution functions are specified in Table 5. The results of the mean values of H(z) and the standard deviation σ_H are listed in table 6. The Histogram plot of H(z) exhibits Gaussian form.

Table 3. Distribution ranges of the model parameters

| Model Parameters	Distribution Range
H_o	$\sim U(65.2, 70.8)$
Ω_{m0}	$\sim U(0,1)$
$\Omega_{\lambda 0}$	$\sim U(0,1)$

**Figure 5. A Histogram plot of H (0.24) obtained from
Monte Carlo simulation, with uniformly distributed H_o**

**Table 4. Mean values of H(z) and the standard deviation σ_H
obtained from the Monte Carlo simulation.**

Redshift(z)	Mean H(z)	σ_H
0.1	70.983	0.1001
0.17	74.9119	0.0076
0.24	79.0545	0.0691
0.27	81.0560	0.0333
0.4	89.6971	0.0368
0.43	91.6215	0.0672
0.48	95.1465	0.0178
0.88	126.7464	0.0583
0.9	128.4690	0.2940
1.3	164.7694	0.2096
1.43	178.4251	0.1092
1.53	188.5052	0.0584
1.75	211.8644	0.03172

Table 5. Distribution ranges of the model parameters

Model Parameters	Distribution Range
H_o	$\sim N(68, 2.8)$
Ω_{m0}	$\sim U(0,1)$
$\Omega_{\lambda 0}$	$\sim U(0,1)$
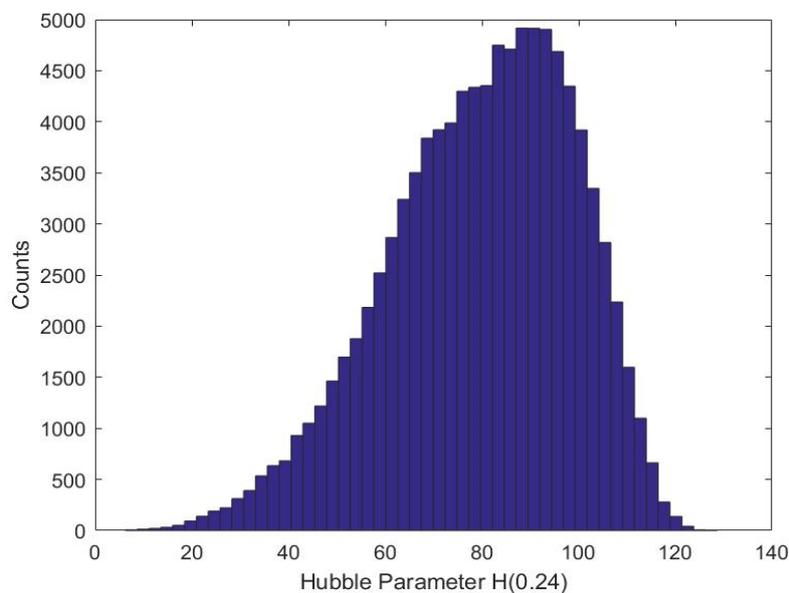
**Figure 6. A Histogram plot of H (0.24) obtained from
Monte Carlo simulation, with normally distributed H_o**

**Table 5. Mean values of H(z) and the standard deviation σ_H
obtained from the Monte Carlo simulation.**

Redshift(z)	Mean H(z)	σ_H
0.1	71.5746	0.0404
0.17	75.5600	0.0670
0.24	79.8412	0.6530
0.27	81.6434	0.0428
0.4	90.1633	0.0584
0.43	92.3130	0.0603
0.48	95.9455	0.0449
0.88	127.6350	0.2266
0.9	129.3052	0.1121
1.3	166.4228	0.3112
1.43	179.6818	0.0617
1.53	189.6652	0.4226
1.75	213.3414	0.0581

4. Conclusion

The 13 magnitude-redshift dataset is analyzed and the χ^2 test yielded a minimum at 8.1015 which corresponds to the model parameters $\Omega_{m,0} = 0.31, \Omega_{\lambda,0} = 0.78$ .The constraint contours are plotted in the around the best fit values. The likelihood test yielded a slightly different values of the parameters. The most likely values obtained from the likelihood functions are $\Omega_{m,0} = 0.28, \Omega_{\lambda,0} = 0.72$. The results of two estimates largely agree. The Monte Carlo simulation was performed over 100000 samples for each of the 13 magnitude-redshift data and the frequency distribution of the Hubble Parameter was found to exhibit Gaussian character. Mean and the Standard Deviation of the results so obtained are also reported. The two-component spatially flat λCDM model provides good fit to the experimentally observed data.

5. References

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