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Abstract: *Given this, it is found that the exchange phase for polarized fermions is provided by the Berry phase. In the case of noncontractible loops in a nonsimply connected space, the exchange phase lacks curvature and is associated with the first Stiefel-Whitney class of the two spin bundle. In contrast, the Chern class—where the Chern number is determined by the integral of the curvature—is linked to the Berry phase, which is the holonomy element of the $U(1)$ connection. The exchange phase for polarized fermions is determined by the Berry phase. This is because the Z_2 -symmetry breakdown in this scenario means that the Stiefel-Whitney class, which is significant in the Z_2-cohomology, is not applicable. Because of this, the exchange phase for chiral fermions may be realized as the well-known Berry phase.*

1 Introduction

For noncontractible loops in a nonsimply linked space, the exchange of two identical fermions is related to the first Stiefel-Whitney class of the two spin bundle. This is due to the fact that the manifold's orientability is obstructed by the action of exchanging the locations of two spins. We will attempt to tackle the issue in this note by approaching it via the lens of a fermion's quantization within the context of Nelson's stochastic mechanics [1]. It was noted in a previous study [2–3] that the introduction of an internal variable that functions as a direction vector can accomplish both the quantization of a fermion and the relativistic generalization of Nelson's stochastic mechanics.The internal degrees of freedom that constitute spin are really depicted by this direction vector. An $SU(2)$ gauge bundle can be used in this approach to describe a spin. This facilitates the recasting of the conveyed spin basis in terms of gauge currents using the Berry-Robbins formalism. The exchange phase is compatible with the first Stiefel-Whitney class and may be understood in terms of a geometric phase without any curvature. On the other hand, we may associate the exchange phase with the well-known geometric phase—the Berry Phase—for a polarized fermion. In fact, the only fermions that exist for a chiral spinor are left- or right-handed ones with a certain spin polarization that requires the breakdown of the reflection symmetry (Z_2 -symmetry). By doing this, we are able to stay away from the Stiefel-Whitney class, which is valuable in Z_2- cohomology. It is possible to think of a polarized fermion as one that lacks a spin degree of freedom. In this instance, the Berry phase may be used to understand the exchange phase. It should be mentioned that this phase is linked to chiral anomaly, which is caused by chiral symmetry breakdown [4]. We will examine the situation involving polarized fermions, and it is demonstrated that the exchange phase in this scenario is connected to the well-known Berry phase.

2. Theoretical Background

The scenario where the Dirac spinor is divided into chiral form and the two helicities match up with left- and right-handed spinors is examined here. It is commonly known that chiral anomaly results from the violation of chiral symmetry.

In this case the divergence of the topological current [5] $\vec{J}_{\mu}$ given by eqn.

$$\vec{J}_{\mu} = \varepsilon^{\mu\nu\lambda\sigma}\left(\vec{a}_{\nu} \times \vec{f}_{\lambda\sigma}\right) = \varepsilon^{\mu\nu\lambda\sigma}\partial_{\nu}\vec{f}_{\lambda\sigma} \qquad (1)$$

does not vanish. In fact it satisfies the relation [6]

$$\left(\partial_{\mu} J_{\mu}^{a}\right) \neq 0 \qquad (1)$$

where The second part of the gauge current is denoted by J_{μ}^{2}, while the axial vector current $\bar{\psi}\gamma_{\mu}\gamma_{5}\psi$ linked to the chiral spinor field is represented by J_{μ}^{5}. In addition, the three gauge current components fulfill the relation.

$$J_{\mu}^{1} = -\left(\frac{1}{2}\right) J_{\mu}^{2} \; ; \quad J_{\mu}^{3} = \left(\frac{1}{2}\right) J_{\mu}^{2} \qquad (2)$$

It is noted that when the divergence of the current J_{μ}^{a} (a=1 ,2, 3 corresponding to the group index) does not vanish then the parallel transport rule given by eqn.(3)

The relation governs the parallel transit rule for the exchange of spins

$$\left\langle M'(\vec{r}) \middle| \nabla M(\vec{r}) \right\rangle = 0 \qquad (3)$$

implying that there is no usual local gauge potential in this case which gives rise to the familiar geometric phase (Berry phase).

Eqn.(3) is modified and satisfies the relation

$$\left\langle M'(\vec{r}) \middle| \nabla \left\{ M'(\vec{r}) \right\} \right\rangle \neq 0 \qquad (4)$$

where the gauge current is used to characterize the spin state. Indeed the relation (4) follows from the condition $\left(\partial_\mu J_\mu^a\right) \neq 0$. This basically suggests that there is an $U(1)$ gauge field, and the holonomy component of this relationship is the Berry phase. It is observed that the divergence of the axial vector current is given by [6-9] when a chiral spinor interacts with a gauge field.

$$\partial_\mu J_\mu^5 = \left(\frac{1}{8\pi^2}\right) Tr\, {}^*F_{\mu\nu} F_{\mu\nu} \tag{5}$$

where ${}^*F_{\mu\nu}$ is the Hodge dual

$${}^*F_{\mu\nu} = \left(\frac{1}{2}\right) \varepsilon_{\mu\nu\lambda\sigma} F^{(\lambda\sigma)} \tag{6}$$

So from the relation (1) we have

$$\partial_\mu J_\mu^2 = -\left(\frac{1}{16\pi^2}\right) Tr\, {}^*F_{\mu\nu} F_{\mu\nu} \tag{7}$$

and from the eqn.(2) we note that $\partial_\mu J_\mu^1$ and $\partial_\mu J_\mu^3$ can be related with this term. It is noted that the topological Lagrangian and the gauge current's divergence, as provided by equation (7), are connected

$$L = -\left(\frac{1}{4}\right) Tr\, \varepsilon^{\mu\nu\lambda\sigma} F_{\mu\nu} F_{\lambda\sigma} \tag{8}$$

In fact, the primary geometrical characteristic underlying this topological term appears to be the anisotropic feature connected to the "direction vector," which is in charge of the quantization of a fermion in 3 + 1 dimensions [10–12]. This anisotropic property for chiral fermions leads to chiral anomaly linked with Berry phase. If the topological term provided by equation (8) is included in the Lagrangian associated with a non-Abelian gauge theory, it will create an Abelian background gauge field that is topologically nontrivial in the configuration space. Actually, in 3 + 1 dimensionsional space-time, this term results in a vortex line in gauge orbit space [13–14].

In a non-Abelian gauge theory, the hidden Abelian gauge field associated with the vortex line can be understood as if, in the gauge orbit space, a particle denoted by A (a non-Abelian gauge potential) moves in the space U of non-Abelian gauge potentials under the action of an Abelian gauge field.

In the language of differential forms we can write

$$A = \left(g^{-1}dg\right) + \left(g^{-1}ag\right) \qquad (9)$$

where $A = \left(A_\mu dx^\mu\right)$ and $a = \left(a_\mu dx^\mu\right)$.

The space of gauge orbit G stands for the space of local gauge alterations in U/G. The points in $a(x)$ make up $g(x)$. One may note that U is contractible as it is the space of all non-Abelian gauge potentials.

Noting that $\{\pi_3(G)\} = Z$ all simple non-Abelian groups G and $\{\pi_2(G)\} = Z$, $\{\pi_n(U)\} = 0$ for all, n we have

$$\pi_n(U/G) = \{\pi_{n-1}(G)\} \quad n\rangle 1 \qquad (10)$$

In 3 + 1 dimensions we have

$$\pi_1(U/G) = \{\pi_0(G)\} = \{\pi_3(G)\} = Z \qquad (11)$$

In fact when we take the x-space as a compactified three sphere S^3, $g(x)$ defines a map $S^3 \rightarrow G$.

The requirement that the gauge transformation $g(x)$ approaches a constant regardless of the direction of x as $x \rightarrow \alpha$ leads to the equality $\pi_0(G) = \pi_3(G) = (Z)$. Hence, U/G is ring-topped and multiple linked, suggesting the presence of a vortex line that is topologically comparable to a magnetic flux line. It has recently been shown that when a loop may be seen as an orbit, a gauge orbit space effectively depicts a loop space [15]. As a result, the foregoing conclusion implies that a magnetic flux line is surrounded by the loop when the topological term provided by equation (8) is included in the theory.When a scalar particle encircles the loop enclosing the magnetic flux line, the particle acquires a geometric phase (Berry phase) given by $e^{i2\pi\mu}$, μ being the monopole strength and $\mu = 1/2$ corresponds to one magnetic flux line [4]. Thus, when the particle passes through a closed circuit that represents a 2π rotation, it gets the phase $e^{i\pi}$, which is the phase associated with a fermion. It appears that the system is a fermion, and the direction of the magnetic flux line indicates whether the spin state is up or down. Therefore, the scalar particle around the loop represents a polarized fermion when it surrounds a magnetic flux line with a certain orientation.Now we note that when a polarized fermion is described by a scalar particle moving around a magnetic flux line with a specific orientation attached to a point $\vec{r}_1$ inside one loop and for another fermion the magnetic flux line having same orientation is attached to the point $\vec{r}_2$ inside the other loop, we may view it as if two spins are attached to the points $\vec{r}_1$ and $\vec{r}_2$. If Feynman [16] defines the exchange of two spins as the rotation of both particles in a half circle, then the rotation of either particle in a whole circle when it encloses a single magnetic flux line describes this equivalently. The result of this will be the geometric phase $e^{i2\pi\mu}$ with $\mu = 1/2$. The Berry phase therefore shows the exchange phase, which occurs when two spins swap locations. It should be noted that μ in the spherical harmonics $Y_l^{m,\mu}$ is connected to the angle χ, which stands for the direction vector ξ_μ rotational orientation. The presentation of this perspective The expansion of the canonical system with a certain internal structure is efficiently handled by χ. This increases the configuration space $S^2 \to S^3$ in compactified space, and the angle χ represents Hopf fibration of S^2 by acting as a gauge degree of freedom similar to $U(1)$. $e^{-i\mu\chi}$ provides [17] the angular component of the spherical harmonics related to the angle χ. So from the relation

$$i\left(\frac{\partial}{\partial\chi}e^{-i\mu\chi}\right) = \mu e^{-i\mu\chi} \qquad (12)$$

we note that when χ is changed to $\chi + \partial\chi$ we have the relation

$$i\left\{\frac{\partial}{\partial(\chi+\partial\chi)}e^{-i\mu\chi}\right\} = i\left\{\frac{\partial}{\partial(\chi+\partial\chi)}e^{-i\mu(\chi+\partial\chi)}e^{i\mu\delta\chi}\right\} \qquad (13)$$

This implies that the wave function will acquire an extra factor $e^{i\mu\delta\chi}$ due to the infinitesimal change of the angle. For one complete rotation the phase is

$$\left(e^{i\mu}\right)\int_0^{2\pi}\delta\chi = e^{i2\pi\mu} \qquad (14)$$

which is the Berry phase [4]. For exchange rotation when both particles traverse a half-circle, the associated phase is given by

$$\left(e^{i\mu}\right)\int_0^{\pi}\delta\chi = e^{i\pi\mu} \qquad (15)$$

Therefore, the phase is $e^{i\pi}$ for polarized fermions when the matching magnetic flux lines have the same orientation, suggesting $\mu = \pm 1$. This is equivalent to the rotation in a full circle by either particle encircling one magnetic flux line which generates the geometric phase

$$\left(e^{i2\pi\mu}\right)(\mu = 1/2) = e^{i\pi} \qquad (16)$$

It is discovered that this exchange phase is an example of the well-known Berry phase. It should be noted that the parallel transport rule (3) is not related to any local gauge field since the exchange phase of two spins is linked to the first Stiefel-Whitney class, which does not include any curvature. However the Berry phase is supplied by the holonomy in a Hermitian line bundle which involves the first Chern class and the Chern number is given by the integral of the curvature. Now, considering that the Z_2 symmetry breaks down for a polarized fermion, which is connected to the chiral symmetry breaking, we may understand the connection we have found here for the exchange phase of two polarized spins in terms of the Berry phase. Because of this, the Stiefel-Whitney class, which is connected to the cohomology group with Z_2 coefficients, won't apply in this situation. In fact, for chiral fermions, the Z_2 symmetry breaking corresponds to chiral symmetry breaking that results in chiral anomaly, which is connected to the Berry phase.

3. Conclusion

We have reached the exchange phase for polarized fermions, which makes sense in terms of the Berry phase as chiral symmetry breaking in this scenario results from -symmetry breaking. This is not significant in this instance since the Stiefel-Whitney class is related to the cohomology group by coefficients. Ultimately, the image of a spin that results from a fermion's quantization process may indicate that a fermion is best described as a scalar particle connected by a magnetic flux line. For a large fermion, this facilitates the conceptualization of a Skyrmion (soliton) [17–18].

4 Discussion

A charged particle is observed to adopt the well-known Aharanov-Bohm (A-B) phase in the presence of a magnetic flux tube functioning as a "external" system. The magnetic flux line, included as an internal degree of freedom in this solitonic model of a fermion, generates the Berry phase associated with the chiral anomaly. The relationship between the Berry phase and statistical phase for polarized fermions implies a relationship between the A-B phase, Berry phase, and statistical phase.

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