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ABSTRACT: We study gravity in the context of the non-commutative manifold $M_4 \times Z_2$, where the discrete space Z_2 represents a direction-vector associated with a space-time point. The pertinent geometry involves the non-commutativity parameter provided by the field strength associated with $SL(2,C)$ gauge fields and is dependent on the phase space variables. The topological current associated with the gauge fields produces torsion when general relativity takes the form of teleparallel gravity. This inevitably leads to Weitzenbock geometry, which creates the deformed space-time.

KEYWORDS: Non-Commutative Manifold, Gauge Fields, Teleparallel Gravity, Weitzenbock Geometry, Torsion

1 Introduction

In recent times several authors [1 - 8] have discussed gravity in the framework of non-commutative geometry. Chamseddine [9] has analysed the $SL(2,C)$ formulation of gravity and its extension in non-commutative geometry when the group structure is extended to $GL(2,C)$. It has been shown that the conventional star product formulation involving the non-commutativity parameter $\theta^{\mu\nu}$ as a constant matrix leads to a complex vierbein. This deformation of general relativity gives rise to apart from the massless graviton, a massive graviton and a scalar field. On the other hand, Langmann and Szabo [10] have shown that the non-commutative $U(1)$ gauge theory in flat space leads to contorted space-time when general relativity appears as teleparallel gravity. The $SL(2,C)$ gauge theory of gravity in the non-commutative manifold $M_4 \times Z_2$, where Z_2 is a direction vector connected to a space-time point rather than a two-point space, will be examined in this note. Note that the manifold $M_4 \times Z_2$ may be seen in the context of commutative space-time so that, in addition to the four-dimensional space-time M_4, the direction vector associated with the discrete space Z_2 appears as an additional dimension. However, the discrete space is embedded in the manifold M_4 in the non-commutative space-time framework so that the space-time coordinates fulfill a commutation relation of the kind $[x^\mu, x^\nu] = i\theta^{\mu\nu}$. In fact, we may represent a direction vector associated to a space-time point as a spin, with opposing orientations of the vector resulting in opposing helicities that correspond to states of spin up and down. Discreteness embedded in the space-time manifold M_4 corresponds to a lattice where Lorentz symmetry is violated, integrated through the non-commutativity of space-time coordinates, resulting in an anisotropic space. It was demonstrated in a previous publication [11] that the manifold $M_4 \times Z_2$ leads to the $SL(2,C)$ gauge theory of gravity and to the torsion represented by the Einstein-Cartan action in the context of commutative Riemann space-time. The torsion term is connected to the direction vector linked to the discrete space Z_2, whereas the Einstein curvature is connected to the Riemannian space. In sec.a we consider certain mathematical formalism related to the noncommutative manifold $M_4 \times Z_2$ when Z_2 is not a two-point space but corresponds to a direction vector attached to a space-time point. In sec.b we shall show how this leads to contorted space-time leading to teleparallel gravity.

2. Theoretical Background

a. Manifold $M_4 \times Z_2$ that is non-commutative

It was demonstrated in a previous study [12] that when discrete space-time is included as an internal variable, the manifold $M_4 \times Z_2$ causes a fermion to quantize. We may think of the complexified space-time as having a coordinate system defined by $Z_\mu = x_\mu + i\zeta_\mu$, where ζ_μ denotes a direction vector that is connected to the space-time point x_μ. Fermion and antifermion-corresponding internal helicities result from the two orientations of the direction vector. When the direction vector is connected to θ via the connection relation $\zeta_\mu = \frac{1}{2}\lambda_\mu^\alpha \theta_\alpha (\alpha = 1,2)$, the complex space-time displaying the internal helicity states may be expressed in terms of a two-component spinorial variable $\theta(\bar{\theta})$. This helps us to write the equivalent metric as $g_{\mu\nu}(x,\theta,\bar{\theta})$.This metric gives rise to the $SL(2,C)$ gauge theory of gravity along with torsion depicted by Einstein-Cartan action [11]. The noncommutativity of space-time coordinates in this manifold $M_4 \times Z_2$ arises when we specify the coordinates in terms of the $SL(2,C)$ gauge covariant derivatives in momentum space. In real space these covariant derivatives will correspond to momentum. In view of this we can identify the position and momentum operators as

$$Q_\mu = -i\left(\left(\frac{\partial}{\partial p_\mu} + A_\mu(\vec{p})\right) \right)$$

$$P_\mu = -i\left[\left(\frac{\partial}{\partial q_\mu} + A_\mu(\vec{q})\right) \right], \; A_\mu \in SL(2,C) \qquad (1)$$

In this case, p_μ is the conjugate of q_μ, which represents the momentum variable, and q_μ is the space-time point in Minkowski space M_4. Evidently, we observe that commutation relations will result from this

$$[Q_\mu, Q_\nu] = \left[F_{\mu\nu}(p) \right]$$

$$\Omega = \frac{1}{2}(g^{ij} dp_i \Lambda dq_j) \qquad (2)$$

where $F_{\mu\nu}$ denotes the non-Abelian gauge field's field strength. We can see from equation (1) that the phase space's symplectic form is provided by

$$\Omega = \frac{1}{2}(g^{ij} dp_i \Lambda dq_j) \qquad \text{(i,j=1 ,2, 3)} \qquad (3)$$

With

$$[X_i, X_j] = (iF_{ij}(p)) \qquad (4)$$

Where

$$j^{ij} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \qquad (5)$$

corresponds to the standard symplectic structure, while Δ^{ij} represents the curvature tensor that results from the induced vector potential A_μ. The phase space variables $\phi^a (= p^1, p^2, p^3, q^1, q^2, q^3)$ $a = 1....6$ can be changed to $\phi^a - i\hbar A^a(\phi)$ in the nonrelativistic limit by associating a residual Abelian gauge field A_μ with q_i and p_i. In the covariant derivative form of these phase variables, we express the position and momentum variables as

$$X_i = \left(-i \frac{\partial}{\partial p^i} - A_i(p) \right)$$

$$P_i = \left(i \frac{\partial}{\partial p^i} - A_i(q) \right) \qquad (6)$$

We have the commutation relations

$$[X_i, X_j] = i\left\{ F_{ij}(p) \right\}$$

$$[P_i, P_j] = i\left\{ F_{ij}(q) \right\} \qquad (7)$$

For two functions $\phi(X)$ and $\chi(X)$ the star product is defined as

$$\phi(x) * \chi(X) = \left[\exp\left(\frac{i\hbar}{2} \theta_{ij} \partial^i_X \partial^j_Y\right) \phi(X) \chi(Y) \right] \Big|_{Y=X}$$
$$= \phi(X)\chi(X) + \frac{i\hbar}{2} \left\{ \theta_{ij} \partial^i \phi(X) \partial^j \chi(Y) \right\} + HOT \qquad (8)$$

HOT- means higher order terms , where θ_{ij} is the antisymmetric constant matrix given by the usual symplectic structure. The star commutator known as the Moyal bracket is defined as

$$(\hbar = 1) \qquad (9)$$

If we consider the case in which ϕ and χ correspond to space-time coordinates X_i and X_j Then from eqn (8) we find (neglecting the higher order terms)

$$[X_i, X_j] = \left(i\theta_{ij}\right) \qquad (\hbar = 1) \qquad (10)$$

Thus comparing eqns. (7) and (10) we can identify θ_{ij} with the curvature tensor F_{ij} provided F_{ij} is trivial and represents a constant matrix. It is noted that when the curvature F_{ij} is trivial such that $\partial F_{ij} = 0$, F_{ij} becomes a constant matrix and by normalization we can relate F_{ij} with the matrix

$$j^{(ij)} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

in eqn.(4) associated with the usual symplectic structure. In this case the star product of two functions f(Á) and g(Á) , Á being the phase space variable is given by

$$\{(f * g)(\phi)\} + \{f(\phi)g(\phi)\} + \left(\frac{i\hbar F_{ij}}{2}\right)\partial_i f(\phi)\partial_j g(\phi) + O(\hbar^2) \qquad (11)$$

using the typical symplectic matrix denoted by $F_{ij} = \theta_{ij}$. This leads us to conclude that the induced background constant magnetic field may be considered the source of the product deformation. It has been noted that the symplectic structure will deform when F_{ij} is nontrivial, causing the noncommutativity parameter μ_{ij} to become a function of ϕ $(\phi = p, q)$. The Berry phase, which is connected to chiral anomaly, results from this distortion of the symplectic structure [13–18].

b. Noncommutative Geometry, Torsion and Teleparallel Gravity

In order to be thorough, we repeat here some findings from a previous study [11] that we found when we examined the manifold $M_4 \times Z_2$ in the context of the "direction vector" appearing as an additional dimension. Torsion, as represented by the Einstein-Cartan action, and the $SL(2,C)$ gauge theory of gravity follow within the framework of commutative space-time of the manifold. The torsion term is connected to the direction vector linked to the discrete space Z_2, whereas the Einstein curvature is connected to the Riemannian space. This is due to the $SL(2,C)$ invariant's invariance in affine space.

Lagrangian is given by

$$L = \frac{1}{4}\left\{Tr(\varepsilon^{\mu\nu\lambda\sigma} F_{\mu\nu} F_{\lambda\sigma})\right\} \qquad (12)$$

This gives rise to the current density [19]

$$\vec{J}^{\mu}_{\theta} = (\varepsilon^{\mu\nu\lambda\sigma}\vec{A}_{\nu} \times \vec{F}_{\lambda\sigma}) = (\varepsilon^{\mu\nu\lambda\sigma}\partial_{\nu}\vec{F}_{\lambda\sigma}) \qquad (13)$$

The total current in a Riemannian space with metric structure is given by

$$\vec{J}^{\mu} = \left(\vec{J}^{\mu}_{R} + \frac{1}{\chi}\vec{J}^{\mu}_{\theta}\right) \qquad (14)$$

Where $\vec{J}^{\mu}_{R} = \vec{J}^{\mu}_{R}.\vec{n}$, $\vec{n}$ being a unit vector and $\chi = -\left(\frac{8\pi G}{c^4}\right)$ [20]. Here $\vec{J}^{\mu}_{R}$ is the contribution to the conserved current due to energy-momentum tensor given by

$$J^{\mu}{}_{R} = \left(\frac{1}{2}\varepsilon^{\mu\nu\beta\gamma}\right)[R_{\varepsilon\beta\gamma}] \qquad (15)$$

where $R_{\varepsilon\beta\gamma} = R_{\varepsilon\beta\gamma}V^{\delta}$,V^{δ} being an arbitrary vector. The total action now represents Einstein-Cartan formulation of gravity given by curvature with torsion. Indeed we have

$$S = \left(S_1 + S_2\right) = \frac{1}{\kappa^2}\int[\vec{J}^{\mu}_{R}\vec{J}^{\mu}_{R} + \vec{J}^{\mu}_{\theta}\vec{J}^{\mu}_{\theta}]d^4x \qquad (16)$$

Where $S = \left(\frac{1}{\kappa^2}\right)\int\left(\vec{J}^{\mu}_{R}\vec{J}^{\mu}_{R}\right)d^4x = -\left(\frac{1}{\kappa^2}\right)\int \mathrm{Re}\, d^4x$ (17)

R being the scalar curvature with the relation

$$\left(\varepsilon^{\mu\alpha\beta\gamma}\varepsilon_{\mu\alpha\beta\gamma}\right) = -e \qquad (18)$$

and κ is the Planck length.

Again writing

$$\left(\vec{A}_{\nu} \times \vec{F}_{\alpha\beta}\right) = \kappa^2 S_{\nu\alpha\beta}\vec{n} \qquad (19)$$

$\vec{n}$ being a unit vector, The action's second term is provided by

$$S_2 = -\left(\frac{4}{\kappa^2}\right)\int S_{\nu\alpha\beta}S^{\nu\alpha\beta}d^4x \qquad (20)$$

leading to torsion.

This analysis shows that the origin of torsion is caused by the topological current J^{μ}_{θ}, which is provided by eqn (13) and is nonvanishing only when $\partial_{\mu}F_{\mu\nu} \neq 0$. This current is linked to the spinorial axial vector current, as demonstrated in a previous study [20], and the torsion that results from this is equivalent to axial vector torsion.

We have shown that torsion is embedded in the space-time manifold and, as a result, space-time is distorted when the discrete space Z_2 is immersed in the space-time manifold M_4, resulting to the noncommutative geometry where the space-time coordinate is provided by Q_μ in equation (1). This results in the teleparallel gravity of general relativity and the implicit entity of torsion of the noncommutative manifold. It should be emphasized that whereas curvature and torsion in the standard Einstein-Cartan action appear as two distinct things independent of one another, in teleparallel gravity, they reflect the same degrees of freedom and serve as other means to describe gravitation.The torsion degree of freedom in the noncommutative manifold $M_4 \times Z_2$ is the manifestation of gravity.

The Lagrangian in teleparallel gravity is given by [21]

$$L_T = \left(\frac{1}{2l^2}\right)\left\{T^\alpha \wedge^* (-{}^{(1)}T_\alpha + 2\,{}^{(2)}T_\alpha + \frac{1}{2}\,{}^{(3)}T_\alpha)\right\} \tag{21}$$

where ${}^{(1)}T_\alpha$, ${}^{(2)}T_\alpha$ and ${}^{(3)}T_\alpha$ corresponds to tensor, vector (trace) and axial vector components of torsion. Its equivalence with general relativity arises from the geometrical identity

$$\left(-\frac{1}{2}R^{\alpha\beta} \wedge \eta_{\alpha\beta} + \frac{1}{2}R^{\{\}\alpha\beta} \wedge \eta_{\alpha\beta} + l^2 L_T\right) = d(e^\alpha \wedge^* T_\alpha) \tag{22}$$

where $R^{\alpha\beta}$ is the curvature associated with the Einstein-Cartan (EC) space-time, $R^{\{\}\alpha\beta}$ is the curvature in Riemannian space-time, e^α is the local frame and $\eta_{\alpha\beta} = {}^*(e_\alpha \wedge e_\beta)$ From this we note that in Weitzenböck space-time with vanishing Riemann-Cartan space curvature $R^{\alpha\beta} = 0$ the Lagrangian L_T is upto a boundary term equivalent to the Einstein-Hilbert Lagrangian. In the framework of noncommutative geometry the vanishing of the EC space curvature $R^{\alpha\beta}$ follows from the fact that the EC action derives from the commutative Riemannian space when the direction vector associated with the discrete space Z_2 appears as an extra dimension. On the other hand, torsion is ingrained in noncommutative space-time, and as a result, curvature and torsion are related. Indeed, curvature and torsion become dual under teleparallel gravity. With certain coefficients of eigenvalue, the teleparallel gravity incorporates all the elements of torsion. It is important to note that the current formalism only accounts for the axial vector torsion associated with chiral anomaly. Consequently, in this noncommutative manifold $M_4 \times Z_2$, the axial vector component of torsion is the only source of contribution in the lagrangian L_T. This is compatible with the fact that the Dirac Lagrangian with external gravity and torsion can only include the axial vector torsion and that the underlying field of the manifold $M_4 \times Z_2$ is a fermion.

3. Conclusion

Here, we have shown that when gravity is supplied by the teleparallel equivalency of general relativity, the noncommutative manifold $M_4 \times Z_2$, where Z_2 corresponds to a discrete space given by a direction vector linked to a space-time point, gives birth to a twisted space-time. The phase space variable determines the noncommutativity parameter $\theta^{\mu\nu}$, which is determined by the field strength connected to the $SL(2,C)$ gauge fields. In the non-Abelian $SL(2,C)$ gauge theory, the teleparallel equivalency of general relativity manifests as a manifestation of the hidden Abelian gauge field.

4. Discussion

It is worth noting that noncommutative $U(1)$ gauge theory in flat space $R^n \times R^n$ leads, via dimensional reduction, to a theory of gravity in R^n, as demonstrated by Langmann and Szabo [2]. We accomplish teleparallel gravity even though in this instance the noncommutativity parameter $\theta^{\mu\nu}$ is assumed to be a constant matrix. The dimensional reduction formulation of the flat phase space $R^n \times R^n$ is, in fact, the primary feature here. It may be observed that a magnetic monopole is involved when the noncommutativity parameter $\theta^{\mu\nu}$ is thought of as a function of the phase space variable. Note that the existence of an inherent monopole in space is suggested by the presence of a direction vector at the space-time point. In actuality, the direction vector serves as a monopole.

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