PROJECTIVE SYNCHRONIZATION OF TWO IDENTICAL FRACTIONAL ORDER LLC CHAOTIC SYSTEMS

T. Suganya, P. Muthukumar

Department of Mathematics, Gobi Arts & Science College, (Affiliated to Bharathiar University, Coimbatore) Gobichettipalayam – 638 453, Tamil Nadu, India

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T. Suganya[1], P. Muthukumar[2*]

Department of Mathematics, Gobi Arts & Science College,
(Affiliated to Bharathiar University, Coimbatore)
Gobichettipalayam – 638 453, Tamil Nadu, India
[1]tsuganyamaths@gmail.com, [2]muthukumardgl@gmail.com

*Corresponding author

ABSTRACT: *This paper comprehensively investigates the chaotic behavior and projective synchronization of two identical fractional order Lorenz-Lü -Chen (LLC) chaotic systems. The Caputo derivative is used to precisely depict the systems fractional dynamics, allowing for a thorough examination of their complex characteristics. Projective synchronization is effectively achieved by applying non-linear controllers, and the work derives and rigorously proves the necessary and sufficient criteria for synchronization. Extensive numerical simulations validate these criteria, confirming the proposed approach's resilience and reliability.*

KEYWORDS: Chaos; Feedback control; Projective synchronization.

1. INTRODUCTION

Fractional-order systems expand the traditional boundaries of calculus by incorporating derivatives and integrals of non-integer orders [1]. This innovative approach allows for the modeling of systems with inherent memory and hereditary effects, providing a more accurate representation of complex phenomena. It extends the analytical reach beyond classical integer-order calculus, offering fresh perspectives on various phenomena. The edibility and adaptability of fractional calculus [25] make it an important tool for investigating and comprehending systems that display complex, non-linear behaviors. The integration of fractional calculus [19] into dynamical systems has given rise to fractional-order models, uncovering additional layers of complexity in physical, biological, and engineering domains. To date, many fractional order differential systems display chaotic behavior such as the fractional order Chen system[21], the fractional order Lu system [22], the fractional order Lorenz system [23], the fractional order unified system [24] and etc.,

Synchronization is a fundamental concept in chaos theory. The person responsible for the development of it is Pecora-Carroll[2]. Chaos theory's exploration of synchronization is revealing new perspectives on the operation of interconnected systems, be they natural or human-made. The process of two or more systems adjusting a specific motion property to shared behavior as a result of inter-action or connection is known as synchronization. In recent years, numerous control strategies and synchronization approaches have been devised for the synchronization of fractional-order systems, including feedback control [3], fuzzy logic control [4, 5], sliding mode control[6, 7], adaptive control[8], active control[9] and etc. Feedback control techniques [20] utilize real time data regarding a system's performance to modify its behavior, ensuring stability or enhancing its efficiency. Synchronization techniques [18] are fundamental for enabling multiple systems or components to function together effectively and harmoniously. The core goal of synchronization is to align the timing and behavior of these elements, which is critical for achieving peak performance and maintaining stability. There are numerous synchronization techniques in chaos

theory, few examples are complete synchronization [11], Hybrid phase synchronization [12], compound-combination synchronization [13], lag and anticipated synchronization [14], feedback synchronization [15], projective synchronization [20].

Projective Synchronization (PS) is a technique in chaos theory where two chaotic systems achieve synchronization by aligning their trajectories with a constant ratio. Unlike traditional synchronization methods that seek exact alignment, PS ensures that the trajectories of the systems maintain a fixed proportional relationship. Projective synchronization of the fractional order chaotic systems was studied in [26, 27]. This method provides deeper insights into chaotic systems and supports the development of sophisticated control strategies and applications, promoting creativity across various scientific fields. Our objective is to design the controller that guides the controlled fractional order LLC chaotic system to asymptotically match the fractional order LLC chaotic system. This alignment leads to synchronization, where the error between the two systems converges to zero over time.

The paper is organized as follows: In Section 2, some basic definitions, properties, and theorems of fractional calculus are given. System description and control techniques are designed for the fractional order dynamical system in Sections 3 and 4 respectively. In Section 5, synchronization between two fractional order LLC chaotic systems is achieved and represents simulation results. Finally, the conclusion of this paper is drawn in Section 6.

2. PRELIMINARIES

This section presents some basic stability theorems after giving a few basic definitions of fractional differential calculus. Caputo's derivative is preferred due to its traditional initial conditions, which align with conventional mathematical conventions and make it easier to apply in various contexts and equations.

Definition. 2.1 The Caputo fractional derivative is defined as

$$D^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} g^{(n)}(\tau) d\tau$$

where $n = [\alpha] + 1, [\alpha]$ is the integer part of α, D^α is called the α-order Caputo differential operator, $\Gamma(.)$ is the usual gamma function given by $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$.

Theorem. 2.1 The commensurate fractional order system is asymptotically stable if and only if

$$|arg(\lambda)| > \frac{\alpha\pi}{2}$$

where λ is the eigenvalue of the corresponding matrix. In this case, each component of the states decays toward 0 like $t^{-\alpha}$. Further, the system is stable if and only if $|arg(\lambda)| \le \frac{\alpha\pi}{2}$, with those critical eigenvalues satisfying $|arg(\lambda)| = \frac{\alpha\pi}{2}$ having geometric multiplicity of one.

Theorem. 2.2 A necessary condition for the fractional-order chaotic system to remain chaotic is keeping at least one eigenvalue λ in the unstable region. This means

$$\alpha > \frac{2}{\pi} tan^{-1}\left(\frac{|Im(\lambda)|}{Re(\lambda)}\right)$$

3. SYSTEM DESCRIPTION

A chaotic Lorenz-L$\ddot{u}$-Chen[17] system participation in the classical Lorenz family of chaotic attractor systems, which has a shared template and single parameter μ is known to be provided. A fractional order Lorenz family system in the sense of Caputo operator is given by

$$
\begin{aligned}
D_t^\alpha x(t) &= (25\mu+10)(y(t)-x(t)) \\
D_t^\alpha y(t) &= (28-35\mu)x(t)-x(t)z(t)+(29\mu-1)y(t) \qquad (1)\\
D_t^\alpha z(t) &= x(t)y(t)-\left(\frac{\mu+8}{3}\right)z(t)
\end{aligned}
$$

According to Theorem.2.2, the fractional order LLC chaotic system (1) remains chaos when the fractional order $\alpha > 0.91$. For the following ranges of parameter μ, the system behaves as the Lorenz, L$\ddot{u}$ and Chen chaotic systems. If $\mu \grave{o}[0,0.8)$, $\mu=0.8$ and $\mu \grave{o}(0.8,1]$ then we get Lorenz, L$\ddot{u}$ and Chen systems respectively. Different portraits of chaotic attractors of the proposed system (1) are shown in Figures.1--3.
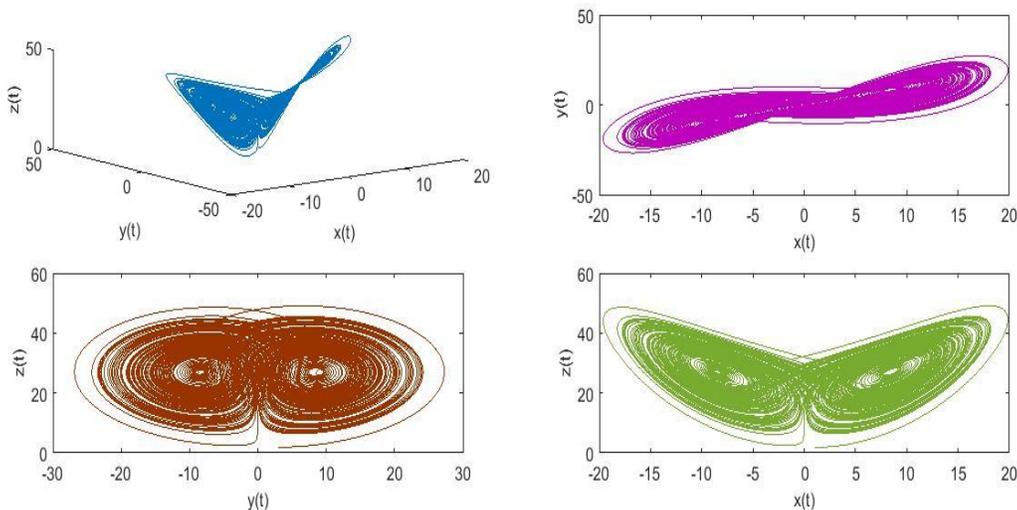

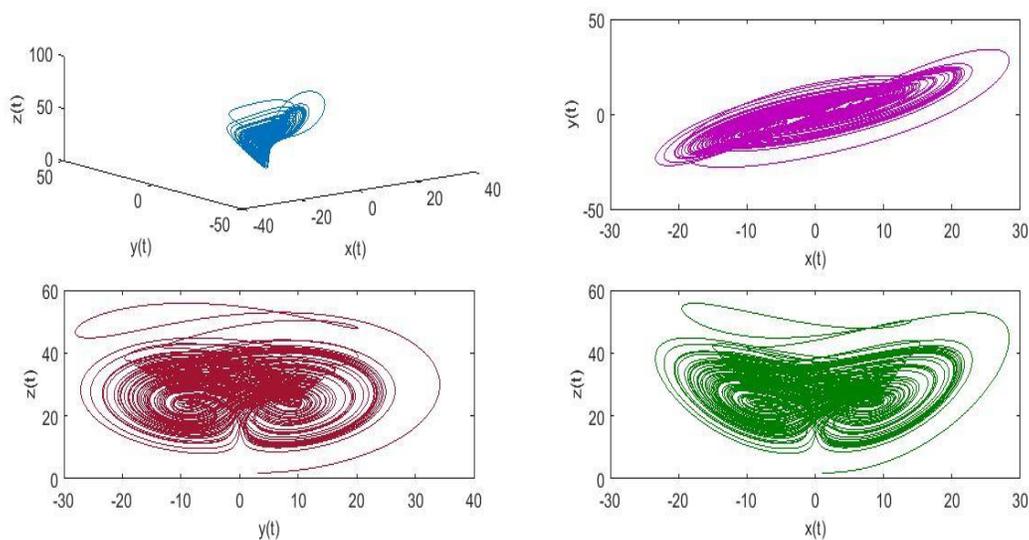Figure.1. Different phase portraits of Lorenz system ($\mu=0$) of (1) when $\alpha=0.97$.

Figure.2. Different phase portraits of L$\ddot{u}$ system ($\mu = 0.8$) of (1) when $\alpha = 0.97$.
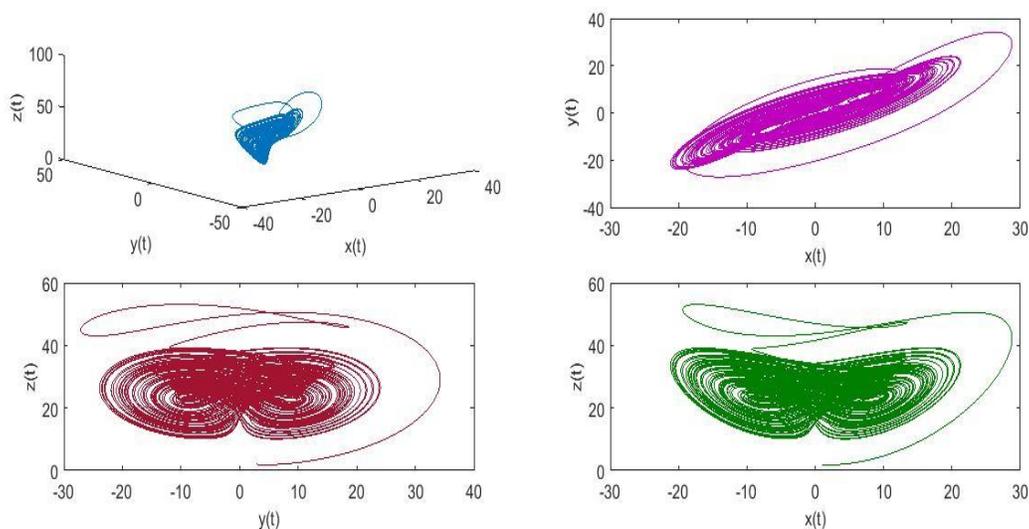

Figure.3. Different phase portraits of Chen system ($\mu = 1$) of (1) when $\alpha = 0.97$.

4. CONTROL OF PROPOSED SYSTEM

In this connection, feedback control technique is used to manage and adjust the behavior of dynamical systems to achieve desired performance.

Consider the controlled system of (1) as

$$D_t^\alpha x(t) = (25\mu + 10)(y(t) - x(t)) + C_1$$
$$D_t^\alpha y(t) = (28 - 35\mu)x(t) - x(t)z(t) + (29\mu - 1)y(t) + C_2 \qquad (2)$$
$$D_t^\alpha z(t) = x(t)y(t) - \left(\frac{\mu + 8}{3}\right)z(t) + C_3$$

where $C_i's, i = 1, 2, 3$ are control signals, which are composed of two distinct elements. The initial element is intended to neutralize the nonlinear aspect of (2). Regarding the controlled fractional LLC system in equation (2), the control signals are determined as

$$C_1 = -(25\mu + 10)y(t) + p_1$$
$$C_2 = x(t)z(t) - (28 - 35\mu)x(t) + p_2$$
$$C_3 = -x(t)y(t) + p_3$$

where $p_1 = -m_1 x(t), p_2 = -m_2 y(t), p_3 = -m_3 z(t)$ are functions as an external input to equation (2). The control design approach involves identifying the optimal values p_i, which functions as gain to achieve stability of (2) Then (2) becomes

$$D_t^\alpha x(t) = -(25\mu + 10)x(t) - m_1 x(t)$$
$$D_t^\alpha y(t) = -(1 - 29\mu)y(t) - m_2 y(t)$$
$$D_t^\alpha z(t) = -\left(\frac{\mu + 8}{3}\right)z(t) - m_3 z(t)$$

According to Theorem.2.1, by accurately adjusting the gain parameter, an unstable eigenvalue can be moved to a stable position. This fine-tuning is vital, as it changes the systems's dynamic behavior from instability to stability.

Consider the Lyapunov candidate function $V = \dfrac{1}{2}(x^2(t) + y^2(t) + z^2(t))$.

The derivative of V is given by

$$D_t^\alpha V = x(t)(-(25\mu + 10)x(t) - m_1 x(t)) + y(t)(-(1 - 29\mu)y(t) - m_2 y(t)) + z(t)(-\frac{(\mu + 8)}{3}z(t) - m_3 z(t))$$

$$= -(25\mu + 10)x^2(t) - m_1 x^2(t) - (1 - 29\mu)y^2(t) - m_2 y^2(t) - \frac{(\mu + 8)}{3}z^2(t) - m_3 z^2(t))$$

$$< 0$$

According to Lyapunov stability theory, the fractional order system (2) attains asymptotically stable for every $m_i > 0, i = 1, 2, 3$.

4.1. Numerical simulations

For Lorenz system, we choose the value of $m_1 = 15, m_2 = 5$ and $m_3 = 6$ at $\mu = 0$. Then the system (2) becomes

$$D_t^\alpha x(t) = -25x(t)$$
$$D_t^\alpha y(t) = -6y(t) \tag{3}$$
$$D_t^\alpha z(t) = -\frac{26}{3}z(t)$$

For Lü system, we choose the value of $m_1 = 20, m_2 = 25$ and $m_3 = 15$ at $\mu = 0.8$. Then the system (2) becomes

$$D_t^\alpha x(t) = -50x(t)$$
$$D_t^\alpha y(t) = -37.2y(t) \tag{4}$$
$$D_t^\alpha z(t) = -17.93z(t)$$

For Chen system, we choose the value of $m_1 = 15, m_2 = 30$ and $m_3 = 11$ at $\mu = 1$. Then the system (2) becomes

$$D_t^\alpha x(t) = -50x(t)$$
$$D_t^\alpha y(t) = -48y(t) \tag{5}$$
$$D_t^\alpha z(t) = -14z(t)$$

The time response of controlled Lorenz system (3), Lü system (4) and Chen system (5) are depicted in Figures 4-6 respectively.
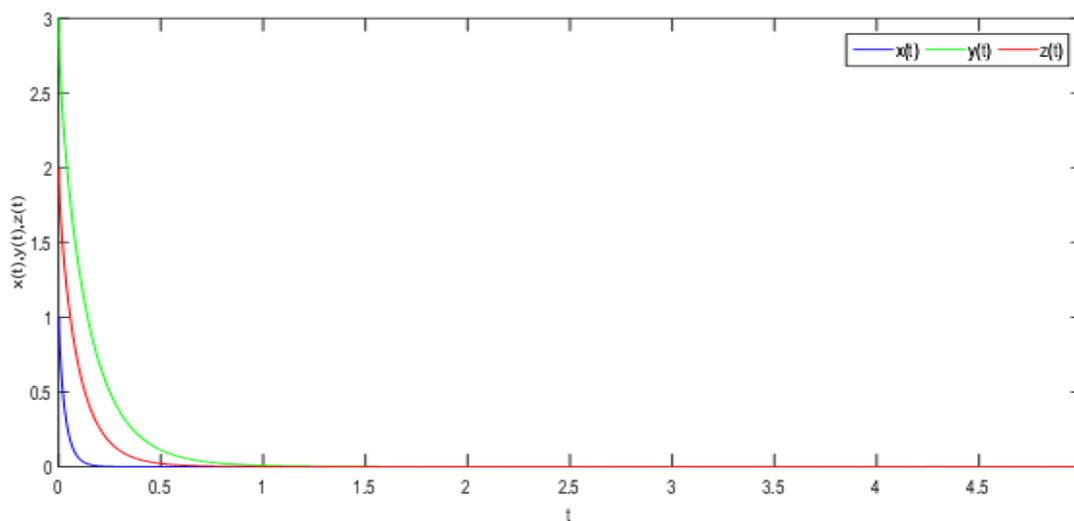

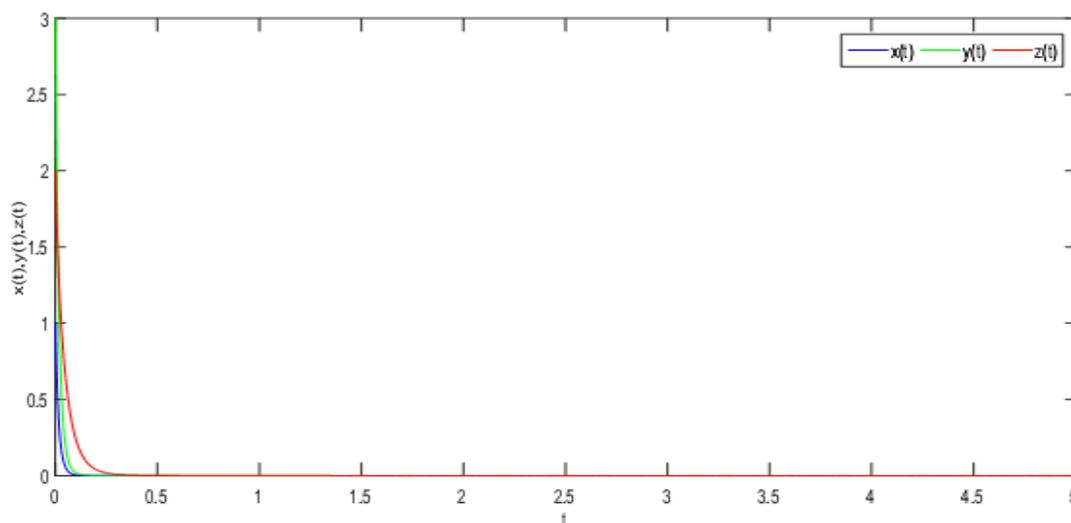Figure.4. Time response of the controlled Lorenz system (3) when $\alpha = 0.97$.

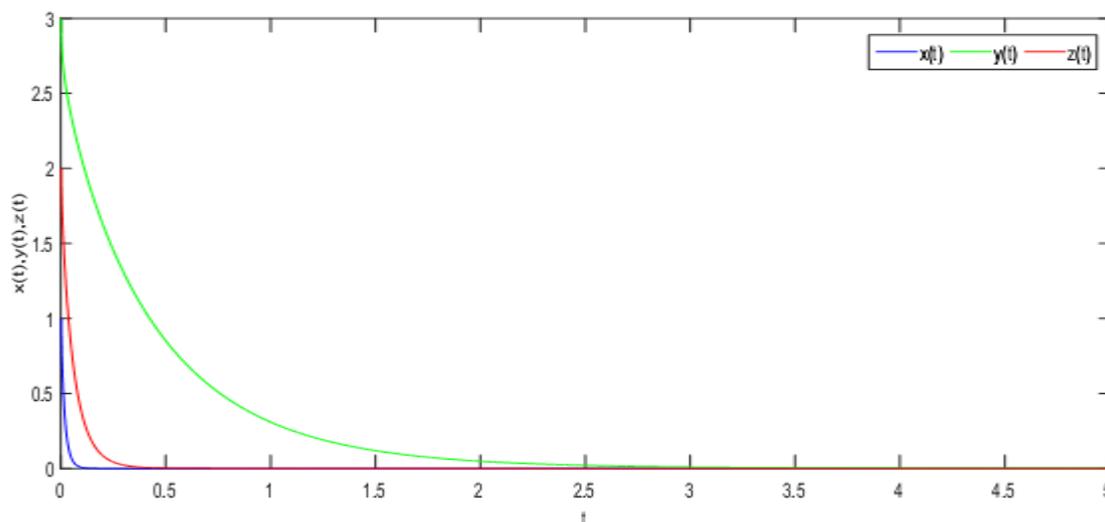Figure.5. Time response of the controlled L$\ddot{u}$ system (4) when $\alpha = 0.97$.

Figure.6. Time response of the controlled Chen system (4) when $\alpha = 0.97$.

5. PROJECTIVE SYNCHRONIZATION OF PROPOSED TWO IDENTICALSYSTEMS

It refers to a synchronization method in chaotic systems where multiple systems align by preserving a constant proportional relationship between their states. Unlike exact synchronization, which requires systems to match their states precisely, projective synchronization allows one system state to be a scaled version of another, governed by a fixed scaling factor.

Consider the fractional order drive system as

$$D_t^\alpha x_d(t) = (25\mu + 10)(y_d(t) - x_d(t))$$

$$D_t^\alpha y_d(t) = (28 - 35\mu)x_d(t) - x_d(t)z_d(t) + (29\mu - 1)y_d(t) \tag{6}$$

$$D_t^\alpha z_d(t) = x_d(t)y_d(t) - \left(\frac{\mu + 8}{3}\right)z_d(t)$$

Consider the response system with controller as

$$D_t^\alpha x_r(t) = (25\mu + 10)(y_r(t) - x_r(t)) + u_1(t)$$

$$D_t^\alpha y_r(t) = (28 - 35\mu)x_r(t) - x_r(t)z_r(t) + (29\mu - 1)y_r(t) + u_2(t) \tag{7}$$

$$D_t^\alpha z_r(t) = x_r(t)y_r(t) - \left(\frac{\mu + 8}{3}\right)z_r(t) + u_3(t)$$

where $u_i(t)$ are controllers $(i = 1, 2, 3.)$ that need to be built in order to achieve projective synchronization between the response and drive systems.

Define projective synchronization errors between the drive and response system as

$$e_1(t) = x_r(t) - \beta_1 x_d(t)$$

$$e_2(t) = y_r(t) - \beta_2 y_d(t)$$

$$e_3(t) = z_r(t) - \beta_3 z_d(t)$$

where β_i's are called non-zero scaling factors. It is very essential for adjusting and aligning the magnitudes of system output to achieve proper control and performance and ensures that two chaotic systems synchronize up to given scaling ratio. Then the fractional order error system is given by

$$D_t^\alpha e_1(t) = D_t^\alpha x_r(t) - \beta_1 D_t^\alpha x_d(t)$$

$$D_t^\alpha e_2(t) = D_t^\alpha y_r(t) - \beta_2 D_t^\alpha y_d(t)$$

$$D_t^\alpha e_3(t) = D_t^\alpha z_r(t) - \beta_3 D_t^\alpha z_d(t)$$

Then

$$D_t^\alpha e_1(t) = -(25\mu + 10)e_1(t) + [(25\mu + 10)(y_r(t) - \beta_1 y_d(t))] + u_1(t)$$

$$D_t^\alpha e_2(t) = (28 - 35\mu)(x_r(t) - \beta_2 x_d(t)) - x_r(t)z_r(t) + (29\mu - 1)e_2(t) - x_d(t)z_d(t) + u_2(t) \tag{8}$$

$$D_t^\alpha e_3(t) = x_r(t)y_r(t) - \beta_3 x_d(t)y_d(t) - \left(\frac{\mu + 8}{3}\right)e_3(t)) + u_3(t)$$

To achieve PS, specific control strategies are implemented. These strategies involve adjusting system parameters or adding control functions to the dynamics of the chaotic systems. The goal is to make the state variables of one system converge to a scaled version of the corresponding variables of the other system at a fast rate.

Theorem. 5.1 For scaling factor β_i and any initial condition, the projective synchronization between two coupled fractional order LLC chaotic systems (6) and (7) is achieved under the control laws:

$$u_1(t) = -(25\mu + 10)[y_r(t) - \beta_1 y_d(t)] - k_1 e_1(t)$$
$$u_2(t) = -(28 - 35\mu)[x_r(t) - \beta_2 x_d(t)] + x_r(t)z_r(t) + x_d(t)z_d(t) - k_2 e_2(t) \qquad (9)$$
$$u_3(t) = \beta_3 x_d(t) y_d(t) - x_r(t) y_r(t) - k_3 e_3(t)$$

where $k_i, i = 1, 2, 3$ are positive feedback gains.

Proof

Substitute the values of $u_i (i = 1, 2, 3)$ in (8), the error system can be written as

$$D_t^\alpha e_1(t) = -(25\mu + 10 + k_1)e_1(t)$$
$$D_t^\alpha e_2(t) = -(1 + k_2 - 29\mu)e_2(t) \qquad (10)$$
$$D_t^\alpha e_3(t) = -\left(\frac{\mu + 8}{3} + k_3\right)e_3(t)$$

Let $K = diag(k_1, k_2, k_3)$. Then fractional order error system (10) can be converted into

$$\begin{pmatrix} D_t^\alpha e_1(t) \\ D_t^\alpha e_2(t) \\ D_t^\alpha e_3(t) \end{pmatrix} = \begin{pmatrix} -(25\mu + 10 + k_1) & 0 & 0 \\ 0 & -(1 + k_2 - 29\mu) & 0 \\ 0 & 0 & -(\frac{(\mu + 8)}{3} + k_3) \end{pmatrix} \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix} \qquad (11)$$

That is

$$D_t^\alpha e_i(t) = (A' - K)e_i(t) \qquad (12)$$

According to Theorem.2.1, the matrix $(A' - K)$ in (12) must satisfy the condition

$|arg(\lambda)| > \dfrac{\alpha\pi}{2}$ for every $i = 1, 2, 3$. Then the system (11) is asymptotically stable. Consider the Lyapunov function $V(e_i(t)) = e_i(t)^T (A' - K)e_i(t), i = 1, 2, 3$.

Case I : For Lorenz system at $\mu = 0$, (11) becomes

$$\begin{pmatrix} D_t^\alpha e_1(t) \\ D_t^\alpha e_2(t) \\ D_t^\alpha e_3(t) \end{pmatrix} = \begin{pmatrix} e_1(t) & e_2(t) & e_3(t) \end{pmatrix} \begin{pmatrix} -(10 + k_1) & 0 & 0 \\ 0 & -(1 + k_2) & 0 \\ 0 & 0 & -(k_3 + 2.67) \end{pmatrix} \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix}$$

The fractional order time derivative of $V(e_i(t))$ is given by

$$D_t^\alpha V(e_i(t)) = -(10 + k_1)e_1^2(t) - (k_2 + 1)e_2^2(t) - (k_3 + 2.67)e_3^2(t)$$
$$< 0, (\forall k_i > 0)$$

CaseII : For Lü system at $\mu = 0.8$, (11) becomes

$$\begin{pmatrix} D_t^\alpha e_1(t) \\ D_t^\alpha e_2(t) \\ D_t^\alpha e_3(t) \end{pmatrix} = \begin{pmatrix} e_1(t) & e_2(t) & e_3(t) \end{pmatrix} \begin{pmatrix} -(30+k_1) & 0 & 0 \\ 0 & -(k_2-22.2) & 0 \\ 0 & 0 & -(k_3+2.933) \end{pmatrix} \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix}$$

The fractional order time derivative of $V(e_i(t))$ is given by

$$D_t^\alpha V e_i(t) = -(30+k_1)e_1^2(t) - (k_2-22.2)e_2^2(t) - (k_3+2.933)e_3^2(t)$$
$$< 0, (\forall k_1 > 0, k_2 > 23, k_3 > 0)$$

CaseIII : For Chen system at $\mu = 1$, (11) becomes

$$\begin{pmatrix} D_t^\alpha e_1(t) \\ D_t^\alpha e_2(t) \\ D_t^\alpha e_3(t) \end{pmatrix} = \begin{pmatrix} e_1(t) & e_2(t) & e_3(t) \end{pmatrix} \begin{pmatrix} -(35+k_1) & 0 & 0 \\ 0 & -(k_2-28) & 0 \\ 0 & 0 & -(k_3+3) \end{pmatrix} \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix}$$

The fractional order time derivative of $V(e_i(t))$ is given by

$$D_t^\alpha V(e_i(t)) = -(35+k_1)e_1^2(t) - (k_2-28)e_2^2(t) - (k_3+3)e_3^2(t)$$
$$< 0, (\forall k_1 > 0, k_2 > 28, k_3 > 0)$$

The fractional order error system in all three cases is globally and asymptotically stable according to Lyapunov stability theory. That is, the evaluation of PS errors $e_i(t), i = 1, 2, 3$ are tends to zero as $t \to \infty$, which implies that the PS between two coupled fractional order LLC chaotic systems (6) and (7) is achieved successfully.

5.1. Numerical simulations

The time response of projective synchronization error states of Lorenz system, Lü system and Chen system are depicted in Figures. 7-9 with positive gains $(k_1 = 5, k_2 = 10, k_3 = 15)$, $(k_1 = 10, k_2 = 25, k_3 = 5)$ and $(k_1 = 10, k_2 = 30, k_3 = 5)$ respectively.

6. CONCLUSIONS

In this work, the fractional order LLC system was modeled using the Caputo derivative. Nonlinear feedback controller has been applied to achieve projective synchronization between two identical proposed fractional order chaotic systems. The necessary and sufficient conditions for synchronization were derived and subsequently verified through detailed numerical simulations.
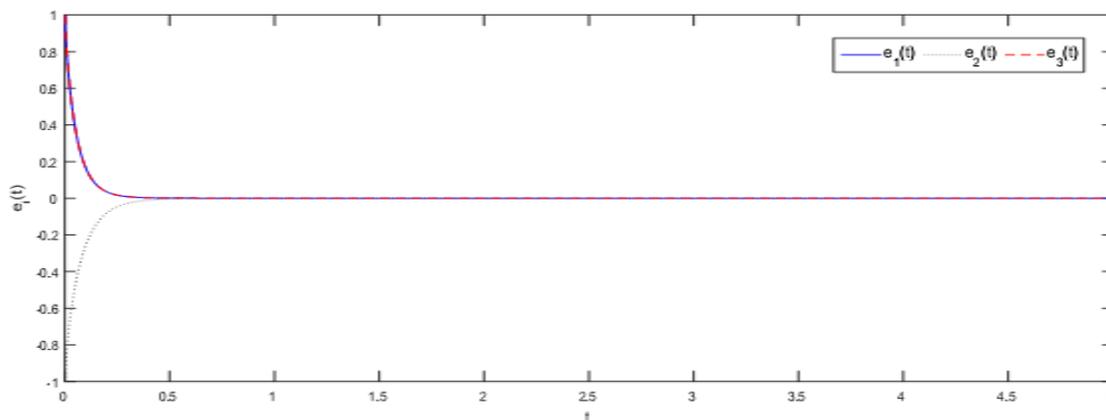
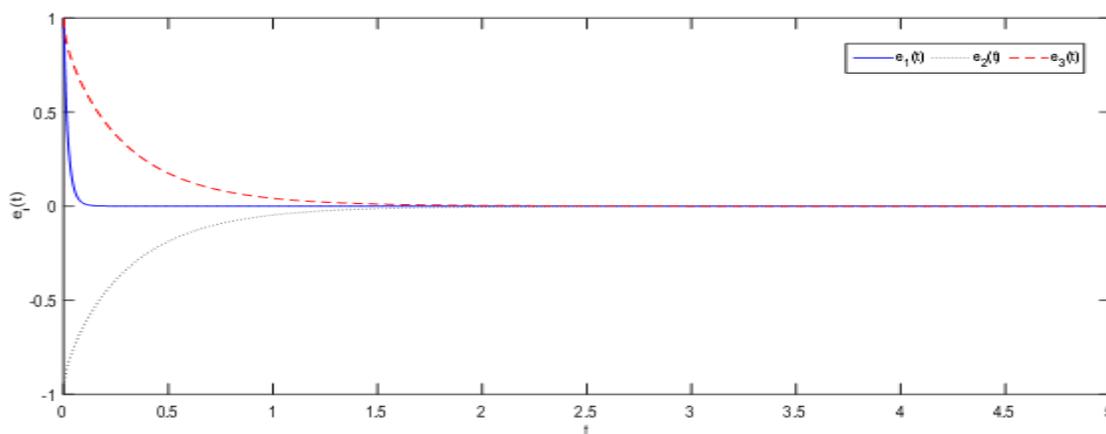Figure. 7. Time response of error states of Lorenz system with $\alpha = 0.97$

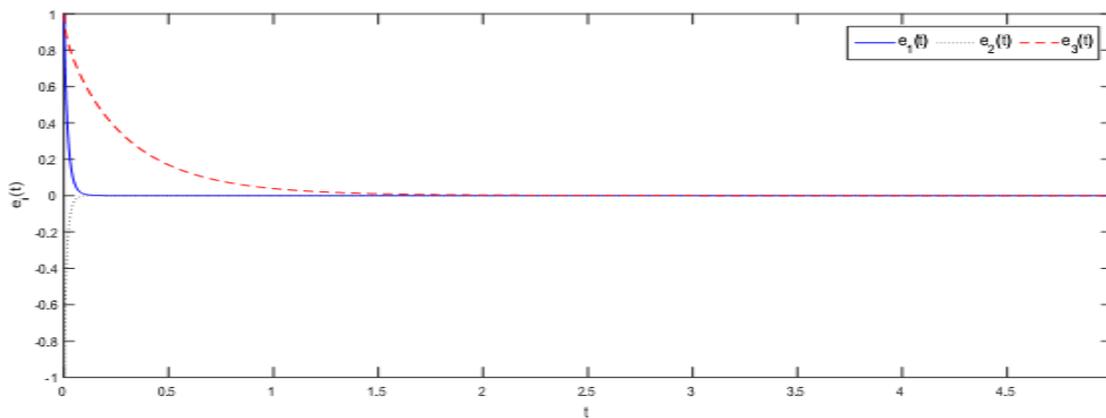Figure. 8. Time response of error states of L$\ddot{u}$ system with $\alpha = 0.97$

Figure. 9. Time response of error states of Chen system with $\alpha = 0.97$

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