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ABSTRACT: *For two vertices v_j and v_k of a graph G, the usual distance D_{jk}, is the length of the shortest path between vertices v_j and v_k. In this paper, we introduce a new concept named D^d Distance energy by considering the degrees of various vertices presented in the path, in addition to the length of the path for $j \neq k$ and zero otherwise. Also, we compute the D^d Distance energy of some standard graphs.*

KEYWORDS: *D^d Distance, D^d Distance matrix, D^d Distance energy, Cocktail party graph, Crown graph, Friendship graph.*

2020 Mathematics Subject Classification*: 05C50, 05C12.*

1. INTRODUCTION

Throughout the paper, all graphs will be assumed to be simple, connected and finite in order to simplify the analysis. There has been a lot of discussion about the concept of Energy of graph since 1978, when Ivan Gutman introduced it. A more detailed explanation of the concept energy can be found in [2,3].

Let G be a simple undirected graph with vertex set $V'(G) = \{v_1, v_2, \dots, v_q\}$. The distance D_{jk} between two vertices v_j and v_k is defined as the length of the shortest path connecting v_j and v_k in a graph G. The degree of a vertex u in G, denoted by $d_G(u)$, represents the number of edges connected to u. Recently, T. Jackuline et al[6] defined the concept D^d Distance of graphs. Motivated by this definition, we introduce a new concept called D^d Distance energy.

We introduce the D^d Distance matrix for a graph G, which is an $q \times q$ matrix $D^d(G) = [M_{jk}]$. The elements of this matrix are defined as follows:

$$M_{jk} = \begin{cases} D_{jk} + d_G(v_j) + d_G(v_k) + d_G(v_j)d_G(v_k), & if \;\; j \neq k \\ 0, & if \;\; j = k \end{cases}$$

The eigenvalues of $D^d(G)$ labeled as $\mu_1, \mu_2, \dots, \mu_q$ are said to be D^d Distance eigenvalues of G, and the set of these values is known as the D^d Distance spectrum of G. The sum of the absolute values of these D^d Distance eigen values is termed as D^d Distance energy of G and is represented as $E_{D^d}(G)$. Mathematically, this is expressed as:

$$E_{D^d}(G) = \sum_{i=1}^{q} |\mu_i|$$

Let I represent the identity matrix. The D^d Distance polynomial of graph G is defined as follows: $P_{D^d}(G,\eta) = \det(\eta I - D^d(G))$. Since, the D^d Distance matrix $D^d(G)$ be a real symmetric with zero trace, the sum of eigen values $\mu_i's$ must equal to zero.

2. BASIC CONCEPTS

Lemma 2.1. [4] If u, v, x and y are all matrices with u non singular, then $\begin{vmatrix} u & v \\ x & y \end{vmatrix} = |u||y - xu^{-1}v|$

Lemma 2.2. [4] If u, v, x and y are all matrices. Let $R = \begin{pmatrix} u & v \\ x & y \end{pmatrix}$ if u and x are commutative matrices, then $|R| = |uy - xv|$

Lemma 2.3. [5] The adjacency matrix $A(K_q)$ of complete graph K_q, then $A^2(K_q) = (q-2)A(K_q) + (q-1)I_q$.

Definition 2.4. [7] Let us take a pair of complete graphs K_q with vertices set $\{p_i, i = 1,2,3, \dots, q\}$ and $\{r_j, j = 1,2,3, \dots, q\}$. Construct a graph by joining p_i to r_i, for $i = 1,2,3, \dots q$ and denote it by $J(K_q^q)$. It has $2q$ vertices and q^2 edges and is q-regular.

3. MAIN RESULTS

D^d Distance Energy of some standard graphs

Definition 3.1. The cocktail party graph CP'_{2q} is the graph consisting of two rows of paired vertices in which all vertices but the paired ones are connected with a graph edge.

Theorem 3.2. Let CP'_{2q} be the cocktail party graph. Then

$$S_p D^d(CP'_{2q}) = \begin{bmatrix} (2q-1)^3 + 1 & 1 - (2q-1)^2 & -[(2q-1)^2 + 1] \\ 1 & q-1 & q \end{bmatrix}$$

and the D^d energy of CP'_{2q} is, $E_{D^d}(CP'_{2q}) = 2(2q-1)^3 + 2$ where $q \geq 2$

Proof : Let the cocktail party graph be CP'_{2q} then it has $2q$ vertices, $2q(q-1)$ edges and is$(2q-2)$ – regular.

Let $\alpha = \{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{2q}\}$ be the eigen values of D^d matrix of CP'_{2q}

Then the D^d matrix of CP'_{2q} has the form $D^d(CP'_{2q}) =$

$$\begin{bmatrix} (2q-1)^2A(K_q) & (2q-1)^2A(K_q)+[(2q-1)^2+1]I_q \\ (2q-1)^2A(K_q)+[(2q-1)^2+1]I_q & (2q-1)^2A(K_q) \end{bmatrix}$$

The corresponding characteristic polynomial $|\alpha I_{2q} - D^d(CP'_{2q})|$ is

$$= \begin{vmatrix} \alpha I_q - (2q-1)^2A(K_q) & -((2q-1)^2A(K_q)+[(2q-1)^2+1]I_q) \\ -((2q-1)^2A(K_q)+[(2q-1)^2+1]I_q) & \alpha I_q - (2q-1)^2A(K_q) \end{vmatrix}$$

$$= \left|\left(\alpha I_q - (2q-1)^2A(K_q)\right)^2 - \left((2q-1)^2A(K_q)+[(2q-1)^2+1]I_q\right)^2\right| \text{ by lemma (2.2)}$$

$$= \left|(\alpha^2 - [(2q-1)^2+1]^2)I_q - 2(2q-1)^2(\alpha+(2q-1)^2+1)A(K_q)\right|$$

$$= (2(2q-1)^2(\alpha+(2q-1)^2+1))^q \left|\frac{\alpha^2-[(2q-1)^2+1]^2}{2(2q-1)^2(\alpha+(2q-1)^2+1)}I_q - A(K_q)\right|$$

$$= (2(2q-1)^2(\alpha+(2q-1)^2+1))^q \left(\frac{\alpha-(2q-1)^2-1}{2(2q-1)^2} - (q-1)\right)\left(\frac{\alpha-(2q-1)^2-1}{2(2q-1)^2}+1\right)^{q-1}$$

$$=(\alpha+(2q-1)^2+1)^q(\alpha+(2q-1)^2-1)^{q-1}(\alpha-(2q-1)^3-1)$$

Hence, the D^d spectrum of CP'_{2q} is

$$S_pD^d(CP'_{2q}) = \begin{bmatrix} (2q-1)^3+1 & 1-(2q-1)^2 & -[(2q-1)^2+1] \\ 1 & q-1 & q \end{bmatrix}$$

and the D^d energy of CP'_{2q} is, $E_{D^d}(CP'_{2q}) = 2(2q-1)^3+2$ where $q \geq 2$

Definition 3.3. The crown graph $H'_{q,q}$ is the graph generated from complete bipartite graph by removing perfect matching.

Theorem 3.4. Let $H'_{q,q}$ be the crown graph. Then, spectrum $S_pD^d(H'_{q,q})$ is

$$\begin{bmatrix} 2q^3-q^2+q+1 & -q^2+q-3 & -(q^2+1)\pm 2 \\ 1 & 1 & each\ (q-1) times \end{bmatrix}$$

and the D^d energy is, $E_{D^d}(H'_{q,q}) = 4q^3-2q^2+2q+2$ where $q \geq 3$

Proof: Let $H'_{q,q}$ be the crown graph then it has $2q$ vertices, $q(q-1)$ edges and is $(q-1)$ - regular.

Let $\eta = \{\eta_1, \eta_2, \eta_3, \dots, \eta_{2q}\}$ be the eigen value. Then the D^d matrix of $H'_{q,q}$ is

$$D^d(H'_{q,q}) = \begin{bmatrix} (q^2+1)A(K_q) & (q^2+2)I_q + q^2A(K_q) \\ (q^2+2)I_q + q^2A(K_q) & (q^2+1)A(K_q) \end{bmatrix}$$

By lemma (2.3), we get the characteristic polynomial is

$$(\eta-(2q^3-q^2+q+1))(\eta-(-q^2+q-3))\left(\eta-(-(q^2+1)\pm 2)\right)^{q-1}$$

Hence, the D^d spectrum $S_pD^d(H'_{q,q}) =$

$$\begin{bmatrix} 2q^3 - q^2 + q + 1 & -q^2 + q - 3 & -(q^2+1) \pm 2 \\ 1 & 1 & each\ (q-1) times \end{bmatrix}$$

and the D^d energy is, $E_{D^d}(H'_{q,q}) = 4q^3 - 2q^2 + 2q + 2$ where $q \geq 3$

Theorem 3.5. The complete bipartite graph $K_{q,q}$ has the spectrum

$$S_pD^d(K_{q,q}) = \begin{bmatrix} -[(q+1)^2+1] & (q-1)[(q+1)^2+1] \pm (q+1)^2 q \\ 2(q-1) & each\ 1\ time \end{bmatrix}$$

and the D^d energy of $K_{q,q}$ is, $E_{D^d}(K_{q,q}) = 4q(q+1)^2 - 2q^2 - 2q - 4$ where $q \geq 2$

Proof : Let the complete bipartite graph be $K_{q,q}$ and the eigen value of D^d

matrix are $\gamma = \{\gamma_1, \gamma_2, \gamma_3, \dots, \gamma_{2q}\}$ respectively. Then the D^d matrix of $K_{q,q}$ is

$$D^d(K_{q,q}) = \begin{bmatrix} [(q+1)^2+1]A(K_q) & (q+1)^2[I_q + A(K_q)] \\ (q+1)^2[I_q + A(K_q)] & [(q+1)^2+1]A(K_q) \end{bmatrix}$$

By lemma (2.3), we get

$$|\gamma I_{2q} - D^d(K_{q,q})| = (\gamma + [(q+1)^2+1])^{2(q-1)}(\gamma - (q-1)[(q+1)^2+1] \pm (q+1)^2 q)$$

Therefore,

$$S_pD^d(K_{q,q}) = \begin{bmatrix} -[(q+1)^2+1] & (q-1)[(q+1)^2+1] \pm (q+1)^2 q \\ 2(q-1) & each\ 1\ time \end{bmatrix}$$

and $E_{D^d}(K_{q,q}) = 4q(q+1)^2 - 2q^2 - 2q - 4$ where $q \geq 2$

Theorem 3.6. Let the complete graph be K_q. Then

$$S_pD^d(K_q) = \begin{bmatrix} -q^2 & q^2(q-1) \\ q-1 & 1 \end{bmatrix}$$

and the D^d energy of K_q is, $E_{D^d}(K_q) = 2q^2(q-1)$ where $q \geq 2$

Proof : Let the eigen value of complete graph K_q be $\beta' = \{\beta'_1, \beta'_2, \beta'_3, \dots, \beta'_q\}$. Then it has q vertices, $\frac{q(q-1)}{2}$ edges and is $(q-1)$ - regular.

Then the D^d matrix of K_q has the form

$$D^d(K_q) = [q^2 A(K_q)]$$

By lemma (2.3), we get

$$(\beta' + q^2)^{q-1}(\beta' - q^2(q-1))$$

Hence,

$$S_p D^d(K_q) = \begin{bmatrix} -q^2 & q^2(q-1) \\ q-1 & 1 \end{bmatrix}$$

and the D^d energy of K_q is, $E_{D^d}(K_q) = 2q^2(q-1)$ where $q \geq 2$

Theorem 3.7. For any $q \geq 1$, the star graph $K_{1,q}$ has the spectrum

$$S_p D^d(K_{1,q}) = \begin{bmatrix} -5 & \frac{5(q-1) \pm \sqrt{16q^3 + 57q^2 - 34q + 25}}{2} \\ q-1 & each\ 1\ time \end{bmatrix}$$

and the D^d energy of $K_{1,q}$ is, $E_{D^d}(K_{1,q}) = 5(q-1) + \sqrt{16q^3 + 57q^2 - 34q + 25}$

Proof : Let us take a star graph $K_{1,q}$ with vertex set $V'(K_{1,q}) = \{v'_0, v'_1, v'_2, \dots, v'_q\}$ where the vertex v'_0 has degree q and the eigen values of D^d matrix are $\mu = \{\mu_0, \mu_1, \dots, \mu_q\}$ respectively.

For this graph, the D^d matrix is

$$D^d(K_{1,q}) = \begin{bmatrix} 0 & 2(q+1) & 2(q+1) & 2(q+1) & \cdots & \cdots & 2(q+1) & 2(q+1) \\ 2(q+1) & 0 & 5 & 5 & \cdots & \cdots & 5 & 5 \\ 2(q+1) & 5 & 0 & 5 & \cdots & \cdots & 5 & 5 \\ 2(q+1) & 5 & 5 & 0 & \cdots & \cdots & 5 & 5 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2(q+1) & 5 & 5 & 5 & \cdots & \cdots & 0 & 5 \\ 2(q+1) & 5 & 5 & 5 & \cdots & \cdots & 5 & 0 \end{bmatrix}$$

On simplification, we get the characteristic polynomial

$$(\mu + 5)^{q-1}(\mu^2 - 5\mu(q-1) - 4q(q+1)^2)$$

and the spectrum would be

$$S_p D^d(K_{1,q}) = \begin{bmatrix} -5 & \frac{5(q-1) \pm \sqrt{16q^3 + 57q^2 - 34q + 25}}{2} \\ q-1 & each\ 1\ time \end{bmatrix}$$

Therefore, $E_{D^d}(K_{1,q}) = 5(q-1) + \sqrt{16q^3 + 57q^2 - 34q + 25}$

Definition 3.8. The friendship graph, denoted by F_q^3, is defined as the graph obtained by taking q copies of the cycle graph C_3 with a vertex in common.

It is clear that $|V(F_q^3)| = 2q + 1$ and $|E(F_q^3)| = 3q$

Theorem 3.9. The spectrum and energy of the friendship graph F_q^3 is

$$S_pD^d(F_q^3) = \begin{bmatrix} -11 & -9 & \dfrac{20q-11 \pm \sqrt{288q^3+688q^2-368q+121}}{2} \\ q-1 & q & each\ 1\ time \end{bmatrix}$$

and $E_{D^d}(F_q^3) = 20q-11+\sqrt{288q^3+688q^2-368q+121}$

Proof: Consider the friendship graph F_q^3 with vertices set $V'(F_q^3) = \{v'_0, v'_1, v'_2, \dots, v'_{2q}\}$ where $\deg(v'_0) = 2q$ and the eigen values of D^d matrix are $\eta' = \{\eta_0, \eta_1, \dots, \eta_{2q}\}$ respectively.

Then the D^d matrix of F_q^3 is

$$D^d(F_q^3) = \begin{bmatrix} 0 & 3(2q+1) & 3(2q+1) & \cdots & \cdots & \cdots & 3(2q+1) & 3(2q+1) \\ 3(2q+1) & 0 & 9 & \cdots & \cdots & \cdots & 10 & 10 \\ 3(2q+1) & 9 & 0 & \cdots & \cdots & \cdots & 10 & 10 \\ 3(2q+1) & 10 & 10 & \cdots & \cdots & \cdots & 10 & 10 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 3(2q+1) & 10 & 10 & \cdots & \cdots & \cdots & 0 & 9 \\ 3(2q+1) & 10 & 10 & \cdots & \cdots & \cdots & 9 & 0 \end{bmatrix}$$

The characteristic polynomial becomes

$$(\eta+11)^{q-1}(\eta+9)^q(\eta^2-(20q-11)\eta-18q(2q+1)^2)$$

$$(\eta+11)^{q-1}(\eta+9)^q\left(\eta-\left(\frac{20q-11 \pm \sqrt{288q^3+688q^2-368q+121}}{2}\right)\right)$$

Hence, the spectrum

$$S_pD^d(F_q^3) = \begin{bmatrix} -11 & -9 & \dfrac{20q-11 \pm \sqrt{288q^3+688q^2-368q+121}}{2} \\ q-1 & q & each\ 1\ time \end{bmatrix}$$

and $E_{D^d}(F_q^3) = 20q-11+\sqrt{288q^3+688q^2-368q+121}$

Theorem 3.10. Let $J(K_q^q)$ be the join of complete graph. Then the D^d Distance spectrum $S_pD^d(J(K_q^q))$ is

$$\begin{bmatrix} (q+1)^2(q-1) \pm \sqrt{q^6+4q^5+8q^4+6q^3-4q+1} & -(q+1)^2 \pm 1 \\ each\ 1\ time & each\ (q-1)\ times \end{bmatrix}$$

and $E_{D^d}(J(K_q^q)) = 2(q+1)^2(q-1) + 2\sqrt{q^6+4q^5+8q^4+6q^3-4q+1}\,; q \geq 2$

Proof: Let $J(K_q^q)$ be the join of complete graph of order $2q,\ q = 2,3,4,\dots,n$ and q^2 edges. Let the eigen value of D^d Distance matrix be $= \{\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_{2q}\}$. Then the D^d Distance matrix of $J(K_q^q)$ has the form

$$D^d\left(J(K_q^q)\right) = \begin{bmatrix} (q+1)^2 A(K_q) & (q+1)^2 I_q + [(q+1)^2+1]A(K_q) \\ (q+1)^2 I_q + [(q+1)^2+1]A(K_q) & (q+1)^2 A(K_q) \end{bmatrix}$$

By lemma (2.3), we get the characteristic polynomial $\left|\lambda I_{2q} - D^d\left(J(K_q^q)\right)\right|$ is

$$= \begin{vmatrix} \lambda I_q - (q+1)^2 A(K_q) & -\left((q+1)^2 I_q + [(q+1)^2+1]A(K_q)\right) \\ -\left((q+1)^2 I_q + [(q+1)^2+1]A(K_q)\right) & \lambda I_q - (q+1)^2 A(K_q) \end{vmatrix}$$

= $\left|(\lambda I_q - (q+1)^2 A(K_q))^2 - ((q+1)^2 I_q + [(q+1)^2+1]A(K_q))^2\right|$ by lemma (2.2)

Simplifying, we get

$$\left(\lambda - ((q+1)^2(q-1) \pm \sqrt{q^6+4q^5+8q^4+6q^3-4q+1})\right)(\lambda + (q+1)^2 - 1)^{q-1}(\lambda + (q+1)^2 + 1)^{q-1}$$

Hence, the D^d Distance spectrum of $J(K_q^q)$ is

$$\begin{bmatrix} (q+1)^2(q-1) \pm \sqrt{q^6+4q^5+8q^4+6q^3-4q+1} & -(q+1)^2 \pm 1 \\ each\ 1\ time & each\ (q-1)\ times \end{bmatrix}$$

and $E_{D^d}(J(K_q^q)) = 2(q+1)^2(q-1) + 2\sqrt{q^6+4q^5+8q^4+6q^3-4q+1}\,;\ q \geq 2$

4. CONCLUSION

Throughout the various fields of chemistry, biology and computer science, there are numerous applications of D^d Distance energy of a graph. Thus, the study of D^d Distance energy remains an active and productive area of research, with implications that go far beyond the domain of graph theory in terms of its applications. In future, we extend this D^d Distance energy to many graph operations, Corona product and Cartesian product of graphs.

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