JOURNAL OF DYNAMICS AND CONTROL

VOLUME 8 ISSUE 9

EDGE BURNING: RELATIONSHIP BETWEEN EDGE AND VERTEX BURNING OF GRID GRAPH

Komala S[1], Mary U[2]

[1]Department of Mathematics, [2]Associate Prof. (Ret.),
Nirmala College for Women (Autonomous), Coimbatore-641018, India

EDGE BURNING: RELATIONSHIP BETWEEN EDGE AND VERTEX BURNING OF GRID GRAPH

Komala S[1], Mary U[2]

[1]Department of Mathematics, [2]Associate Prof. (Ret.),
Nirmala College for Women (Autonomous), Coimbatore-641018, India

Author's mail address: komalaojas1@gmail.com

ABSTRACT: Consider a graph G with the Cartesian product of two Path graphs. In this paper we are concentrating on edge burning for specific Grid graph. The notation of edge burning number [7] of an m × n Cartesian grid graph, denoted as $b^1(G_{m,n})$, the vertex burning number of a grid $b(G_{m,n})$ graph was first studied in [9]. In this paper, we focus exclusively on undirected graphs. Determining the exact edge burning number for an arbitrary grid graph is an NP-hard problem. Burning is a discrete step by step process; the main intention of burning is to acquire a burning number within the limited period of time which is the so-called source of the vertex and We know that a burning number is denoted as b(G). Similarly edge burning is a source of the edge; by using the source edges we can completely burn a graph. The Grid graph is formed by taking the Cartesian product of two Path graphs, denoted by as $P_m \times P_n$. In this paper the main focus is on this graph. Specifically, we studied the edge burning number of the Cartesian product of two path graphs, even in non- symmetric cases (Where the two path graphs have different lengths.) Remarkably, we successfully demonstrated the following cases.

When $m < n$, the vertex burning number $b(G_{m,n})$.

When $m < n$, the edge burning number $b^1(G_{m,n})$.

When $m = n$, the vertex burning number $b(G_{m,n})$.

When $m = n$, the edge burning number $b^1(G_{m,n})$.

When $m > n$, the vertex burning number $b(G_{m,n})$.

When $m > n$, the edge burning number $b^1(G_{m,n})$.

And we are studying among the relationship between edge and vertex burning for the above cases. As well as we are giving the upper bounds for the Edge burning for Grid graph. Edge and vertex burning play crucial roles in ensuring connectivity and resilience within grid graphs. The relationship between these burning numbers provides valuable insights into the structure of such graphs.

KEYWORDS: Grid ($G_{m,n}$), fence, Edge burning, Vertex burning and Total burning.

1. INTRODUCTION

Over the last decade a few researchers have been working on burning. The 'burning Number' concept was introduced by Anthony Bonato in the year of 2014.

To burn a graph, we require a minimum number of edges, which is called as a Burning Number $b^1(G)$ or Source edge number. Burning a graph is a precise procedure; let us fix the fire on any of one of the edges which is identified as the source edge. In the next step, we have to select a source edge among the leftover edges from round one, then the adjacent of the previously selected source edge will begin to burn automatically which is said to be the basic step for attaining a graph. This process will be repeated till all the edges burns subsequently. And our aim is to burn a graph to get a minimum source of edges by burning all the edges of a graph within the limited period of time. There are wide applications on burning number. It is most applied in the field of social media, Networks and the contagion diseases …etc.,

In graph theory, a grid is a network of points (Vertices) connected by lines (called edges) that are arranged in a regular pattern. A grid is a graph composed of evenly spaced horizontal and vertical lines, creating a regular arrangement of squares or rectangles. Grid graphs are essential components in the broader framework of Graph theory. Researchers, engineers, and artists utilize grid graphs for tasks like data plotting, precise drawing, and visualizing mathematical functions. Each square within the grid corresponds to a discrete point on the graph, facilitating accurate measurements and visual representations.

Grid graphs find applications in trend analysis, data analysis, and data retrieval within the stock market. Examples of grids including Excel spreadsheets and Chessboards, where the chessboard serves as a square grid.

2. DEFINITIONS

2.1. Burning

Burning is a discrete time step by step process, the main intention of burning is to acquire a burning number which is so called source of the vertex, within the limited period of time. The burning number is denoted by *b(G).*

2.2. Edge burning

Burning is a discrete time step by step process, the main intention of burning is to acquire a burning number which is so called source of the edge, within the limited period of time. The edge burning number is denoted by $b^1G)$.

2.3. Grid graph

A grid graph, denoted as G(m,n) is formed by taking the Cartesian product of two paths P_m (with m vertices & $m-1$) and P_n(with n vertices & $n-1$). This graph represents a 2-dimensional grid with m rows and n columns. It has a total mn vertices and [n $(m-1)+m(n-1)=2mn-m-n$] edges. Grid graphs are fundamental structures, and from them, we can derive various other graphs, includingP_n, C_n, and squares. Additionally, grid graphs play a role in modeling fractals. [1]

2.4. Fence graph

A fence graph is obtained by grid graph. A grid consists of Cartesian product of G and H denoted by G■H, has edge set *e(G)* ×*e(H)* as well as vertex set *V(G)* ×*V(H).* The size of a fence graph is $|v| \leq |e|$.

2.5 Bounds

In graph theory "bound" refers to the limitations or constraints placed on specific certain parameters or measures associated with a graph. These bounds can apply to various aspects of the graph, such as minimum or maximum number of edges or even specific properties like bipartite graph.

2.6. Square of a grid

The square of a fence graph with vertices labeled as $G_{m,n}$ $(m = n)$ is formed by connecting each vertex. This graph is connected, and its square graph has the same vertex set as the original. The minimum vertex degree in $G_{m,n}$ is 2, while the maximum degree is increased by 4.

3. BURNING FENCES

The prime result of this paper based on the following theorem, and edge burning questions it left open after it was published.

Theorem 3.1. ([6)]). For m = m(n),

$$b(G_{m,n}) = b(P_m \times P_n) = \begin{cases} \sqrt[3]{\frac{3}{2}}\,(1+o(1))\,\sqrt[3]{mn}, & n \geq m = \omega(\sqrt{n}) \\ \theta(\sqrt{n}), & m = O(\sqrt{n})\,. \end{cases}$$

Consider the specific case where ($m =$ n), the resulting in a square grid. Theorem 3.1 provides an asymptotically tight value for the vertex burning b($G_{m,n}$) when n $\geq m = \omega(\sqrt{n})$. However in the remaining cases where m = O($\sqrt{n}$) the growth rate is given differently. Now let us discuss the concept of "Fence". If C>0 we define the grid $G_{c\sqrt{n},n}$ as a fence because it is wider than tall by definition. When we say "Valid" (c> 0), we mean all positive values of c such that c$\sqrt{n}\,\epsilon N$, adhering to the definition of a Cartesian grid. The figure 3. Illustrates an edge burning sequence for the fence figure

Conjecture: 3.2. [3]. If G is a connected graph of order n, then $b(G) \leq \lceil\sqrt{n}\rceil$.

Theorem 3.3. For a path P_n graph edge burning number is $\lceil\sqrt{n-1}\rceil$.

Proof: The line L(G) is a connected graph of G. In the line graph L(G) the vertices would be considered as a edges of G. It is known that, to burn a graph G, minimum two vertices are required. To burn the line graph of a path graph, minimum three vertices are required, hence n ≥ 3. The burning number of a P_n is $\lceil\sqrt{n}\rceil$, and it is evident that the the burning number of L(P_n) is exactly $\lceil\sqrt{n-1}\rceil$, n ≥ 3. So, the burning number of L(P_n) , n ≥ 3, is $\lceil\sqrt{n-1}\rceil$.

The burning number of a L(P_n) graph is exactly$\lceil\sqrt{n-1}\rceil$.

Remark: i) P_1 is a isolated vertex.

ii) $L(P_2)$ is an isolated vertex.

ISSN:1672-6553

Example:

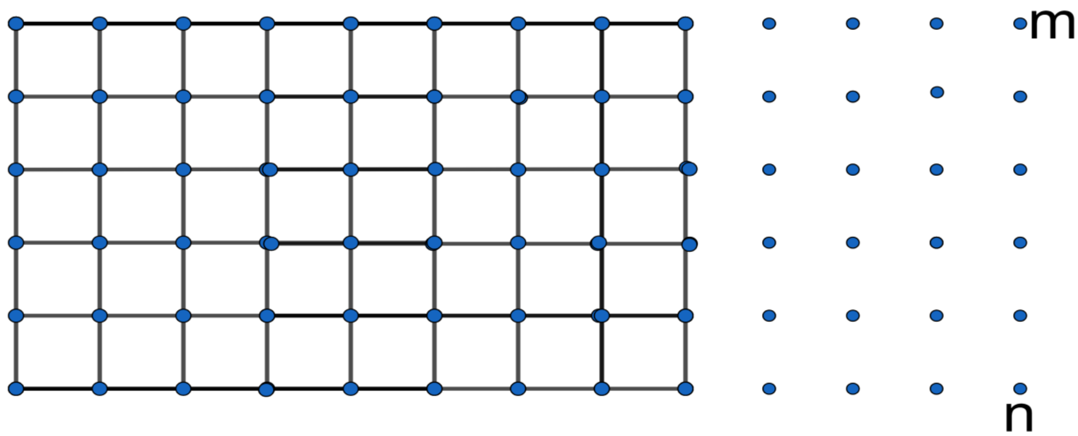

Figure 01. Grid graph.

I. Case

When $m < n$, the vertex burning number *b($G_{m,n}$)*.

In this case we have taken an example $G_{4,n}$, $m < n$

Theorem 3.4. For a fence $G_{m,n}$ graph, If m= 4, b($G_{m,n}$) is $2 + \sqrt{n}$.

Proof: Where $G_{m,n}$ is a fence of a grid, the burning number depends on the value of 'm' and 'n'. We get a precise burning number for fence graph at m= 4. Here 'm' represents the number of 4 rows of a path graph and 'n' represents the number of columns of a path graph. Well we know that, the burning number of a P_n is $\lceil\sqrt{n}\rceil$. So, For a fence the burning number is $2 + \sqrt{n}$ only If m= 4 and n is a perfect square .

$$b(G_{4,n}) = \begin{cases} \sqrt{m} + \sqrt{n}\ , & if\ m = 4\ and\ 'n'\ should\ be\ perfect\ square \\ 0, & if\ m = 1\ and\ n = 1. \end{cases}$$

Remark 3.5. For a fence, $b(G_{4,n})$ is $2 + \sqrt{n}$.This would works only when n is a perfect square otherwise it wouldn't.

II. Case

When m $< n$, the edge burning number b($G_{m,n}$).

In this case we have taken an example $G_{4,n}$, $m < n$

Theorem 3.6. For a fence $G_{m,n}$ graph, If m= 4 then $b^1(G_{m,n})$ is $2 + \sqrt{n}$.

Proof: Where $G_{m,n}$ is a fence of a grid, the edge burning number depends on the value of 'm' and 'n'.We get a precise edge burning number for fence graph at m= 4. Here 'm' represents the number of 4 rows of a path graph and 'n' represents the number of columns of a path graph. Well we know that edge burning number of a path graph is $\lceil\sqrt{n-1}\rceil$. But, For a fence $G_{m,n}$ graph, $b^1(G_{m,n})$ is $2 + \sqrt{n}$ only,If m= 4 and n=$\sqrt{n}$.

$$b^1(G_{4,n}) = \begin{cases} 2+\sqrt{n}\ , & if\ m = 4\ and\ 'n'\ should\ be\ perfect\ square \\ 0, & if\ m = 1\ and\ n = 1. \end{cases}$$

Remark 3.7. For a fence**,** $b^1(G_{4,n})$is $2 + \sqrt{n}$. This would works only when n is a perfect square otherwise it wouldn't.

Conjecture 3.8. For a fence $G_{4,n}$ graph, $b(G_{4,\sqrt{n}}) = b^1(G_{4,\sqrt{n}})$ where 'n' should be perfect square.

Corollary 3.9. $b(G_{4,,\sqrt{n}}) < b(G_{m,n})$.

Result 3.10. After deeply analyzing the case I and case II , we get the following result

1. For $G_{4,n}$, if $m < n$. Certainly the vertex and edge burning number are equal at m=4 and n = $\sqrt{n}$.
2. For a fence $G_{4,n}$, $b(G_{4,\sqrt{n}}) = b^1(G_{4,\sqrt{n}})$

III. Case

When $m = n$, the vertex burning number $b(G_{m,n})$.

In this case we are taking m $= n$, it is a square Grid.

 ISSN:1672-6553

4. VERTEX BURNING OF SQUARE OF A FENCE

Vertex burning is denoted by *b(G)*, where b is the minimum source burning to completely burn a graph**.** Vertex burning number of fence $G_{4,16}$ is 6 [5]. Here we have taken an example of 6× 6 square of a fence. 6× 6 refer to a specific type of structure.

The Maximum and minimum degree lies between, $2 \leq b(G_{m,n}) \leq 4$.

Example

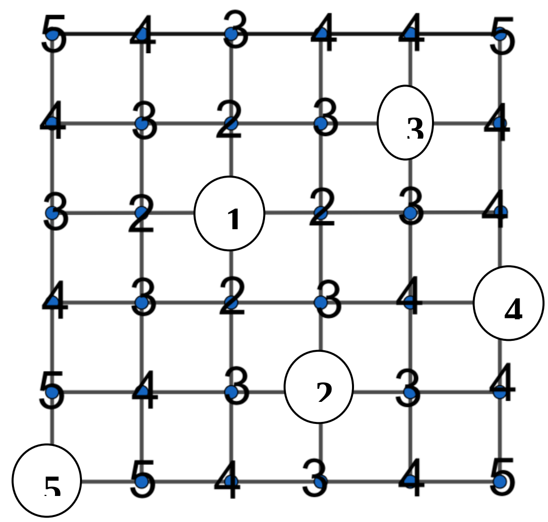

Figure 2. square of a fence

Figure 2: Here, we have a 6×6 square fence, where each vertex is labeled with an index v_i and corresponding to the round in which it serves as the source of the burning. The vertex burning sequence is $(v_1, v_2, v_3, v_4, v_5, v_6)$, where v_i represents the source of the vertex burning. Therefore the $b(G_{6,6})$ is 5.

IV. Case

When $m = n$, the edge burning number $b^1(G_{m,n})$.

In this case we are taking $m = n$, it is a square of a fence.

5. EDGE BURNING OF SQUARE OF A FENCE

Edge burning number is denoted by $b^1(G)$, where $b^1(G)$ is the minimum edge source burning to completely burn a graph**.** Edge burning number of fence$G_{4,16}$ is 6. Here we have taken an example of 6× 6 square of a fence. 6× 6 refer to a specific type of structure.

The Maximum and minimum degree lies between, $2 \leq b^1(G_{m,n}) \leq 4$.

Example

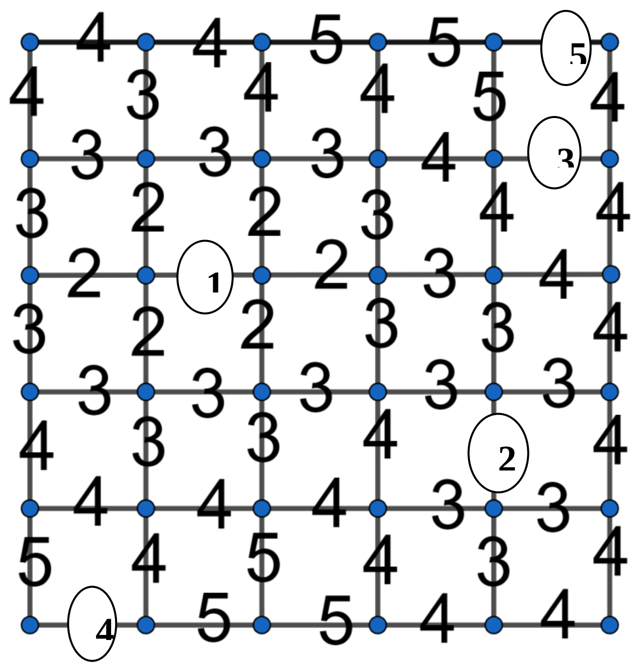

Figure 3. square of a fence

Figure 3. Here, we have a 6×6 square fence, where each edge is labeled with an index e_i corresponding to the round in which it serves as the source of the edge burning. The edge burning sequence is $(e_1, e_2, e_3, e_4, e_5, e_6)$, where e_i represents the source of the edge burning. Therefore the $b^1(G_{6,6})$ is 5.

Theorem 5.1. if $m = n$, is a square of a grid graph $G_{m,n}$ graph. Then, $b(G_{m,n}) = b^1(G_{m,n})$.

Proof: Any square of grid $G_{m,n}$ graph, the number of rows of a path graph is equal to the number of columns of a path graph ($P_m \times P_n$).

Casei. if m $= n = 1$.

Proof : $P_1$is a null graph, the vertex burning number can't be find it out, burning number could be find, only for a connected graph.

Case ii. if $(m = n) > 1$.

Proof: where $(m = n) > 1$, is a square of a grid $G_{m,n}$ graph and m,n $\in v$. The order of a vertex is $|v|$and the order of an edge is$|e|$.

For a grid's square graph $G_{(m=n)>1}$, $|v| \leq |e|$. The burning number and edge burning number are equal to the arbitrary of grid's square graph.

V. Case

When m $> n$, Finding the vertex burning number of $b(G_{m,n})$.

A grid, also known as matrix, is a two – dimensional arrangement organized into $P_m$rows and $P_n$columns. Each cell in the grid contains an element. We can refer to these elements using their row and column indices. Properly specifying the address (dimension) of an element is crucial. Already we have studied the case $m < n$.

In this case we have taken an example $G_{m,4}$, $m > n$, were $m \in N$, and $n = 4$.

If $m = 1$ and $n = 4$ is a P_4 , the burning number of P_4 is 2.

If $m = 2$ and $n = 4$, the burning number of $G_{2,4}$ is 2.

In this case vertex burning of $G_{m,4} = G_{4,n}$, so it means (m $\geq n$) = (m $\leq n$) it holds good particularly for the above case, for reaming cases purely it is an open problem. To justify, If we take a matrix multiplication dimension automatically we understand the logic.

$m \times n$ is equal to the $m \times n$.

VI. Case

When $m > n$, Finding the edge burning number of $b^1(G_{m,n})$. As we studied in the **Case V,** Grid is a $m \times n$ array.

From the conjecture we have fence $G_{4,n}$, $b(G_{4,n}) = b^1(G_{4,n})$.

Result: After studying the V and VI cases, the vertex burning number of $b(G_{m,4})$ and edge burning number of $G_{m,4}$ are same.

$b(G_{4,n}) = b(G_{m,4})$.

6. PARTIAL BURNING

In a graph G, instead of burning all edges, we can focus on a minimum subset of edges denoted by $S \subseteq e(G)$, we define the partial burning number of graph *G* with respect to S as the minimum number of rounds necessary to burn all the edges in S. This parameter is denoted by $b^1(G,S)$. And edge burning number is denoted by $b(G, e(G)) = b^1G)$. Interestingly, we observe that in a 4×n fence graph, $b(G)$ is equivalent to $b^1(G)$. Additionally, in [2], its noted that if *H* is a spanning sub graph of *G* then $b(H) \leq b(G)$. Furthermore, we extend this observation to the setting of partial burning, where adding edges between distinct components of a graph can increase the edge burning number at most once.

7. TOTAL BURNING

Total burning refers to the process of burning vertices or edges in a graph G, where fire spreads along the edges. We define the total burning number of a graph G as b(G) = b(t(G)), representing the minimum number of rounds required for all elements of G to burn. For a connected graph G, there is conjecture

$$b(G) \leq b\big(t(G)\big) \leq b(G) + 1.$$

8. GRID GRAPH EDGE BURNING UPPER BOUNDS

If P_m and P_n are two connected Grid graph, particular for $G_{m,n}$ the edge burning b^1 $(G_{m,n})$ would be min $\{b(P_m) + rad(P_n\ ),\ b(P_n) + rad(P_m)\}$ for some , $m, n \in N$ [3].

Corollary: If $m = o(\sqrt{n})$, then $b^1(G_{m,n}) = (1+o(1))\sqrt{n}$.

Proof: the upper bounds for vertex burning would be min $\{b(P_m) + rad(P_n\ ),\ b(P_n) + rad(P_m)\}$ for some , $m, n \in N$. From the conjecture we have fence $G_{4,n}$, $b(G_{4,n}) = b^1(G_{4,n})$.

So, $b^1(G_{m,n}) \leq b(P_n) + rad(P_m) = \sqrt{n} + o(\sqrt{n}) = (1 + o(1))\sqrt{n}$.

9. DECIMATION

Where an alternative edges removed from a grid graph (whether horizontally, vertically or diagonally), while ensuring that all other edges remain connected, is commonly referred to as decimation. In this modified graph, connectivity is preserved despite the removal of specific edges. Decimation helps optimize data transmission by removing unnecessary edges or links.

9.1. Maximum edges Removed

For $m \times n$ grid graph (with m rows and n columns), total number of edges is given by 2mn-m-n.

To find the maximum of edges can remove without disconnecting the graph[4].

- Start with the total number of edges
- Subtract number of vertices (Which is mn)
- Add 1 to account for the remaining connected component
- The result is the maximum edges you can remove
- Max edges removed = mn-m-n+1

Example: for a 3×3 ($G_{3,3}$) grid, we can remove 4 edges. 3 edges from the center vertex and one edges from an edge surrounding the grid.

Maximum edge removed for ($G_{3,3}$) = 3× 3-3-3+1 = 4.

Maximum edge remove burning is still an open problem.

10. CONCLUSION

We have successfully established upper bounds for edge burning. Specifically, we have shown that for the square grid $b(G_{m,n}) = b^1(G_{m,n})$ at $m = n$. Additionally, we have proved vertex burning is equal to edge burning for certain graphs i.e., $b(G_{4,\sqrt{n}}) = b^1(G_{4,\sqrt{n}})$. And for every square graph $b(G_{m,n}) = b^1(G_{m,n}) \leq \sqrt{mn}$, where $m = n$ Still there are open problems left over in edge burning. The maximum edge removal numbers an open research question. And we can continue to explore these topics to find optimal solutions and bounds.

References

1. Arie Bos , "Index Notation of Grid Graphs", Index Notation of Grid Graphs - arXiv.org .
2. 2. A. Bonato, J. Janssen, E. Roshanbin, "How to burn a graph", *Internet Mathematics 12 (2016)85-100* .
3. Anthony bonato, sean english, bill kay, and daniel moghbel, "Improved bounds for burning fence graphs", . arXiv:1911.01342v1 [math.CO] 4 Nov 2019 .
4. Bret Thacher, "Extensions of external graph theory to grid", *Rose- Hulman undergraduate Mathematics journal,* volume 10, issue2*, 2009.*
5. Daniel Moghbel, "Topics in Graph burning and datalog" , pg no 46.
6. Dieter Mitsche, Pawel Pralat, and Elham Roshanbin, "Burning graphs: a probabilistic perspective" , *Graphs and Combinatorics,* 33(2): 449-471, 2017.
7. Komala & U. Mary, "Edge burning & chromatic burning classification of some graph family", *Journal of Discrete Mathematical Science and Cryptography,* April 2024, 27(2-B):665-674.